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Correspondence between the q-deformed harmonic oscillator and finite range potentials

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The q-deformed harmonic oscillator is revisited in connection with the spectrum of finite range potentials. It is found that the finite series expansion of the canonical variables, the q-deformed phase space coordinate and momentum variables, generates a local momentum dependent interaction. It turns out that the resulting spectrum exhibits features of the spectrum of a finite range potential, added to the low-lying harmonic oscillator behavior. Thus, the otherwise unbounded spectrum of the harmonic oscillator behaves, in the q-deformed version, as the spectrum of a finite range potential subject to specific boundary conditions.

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I. INTRODUCTION

Among the different deformed Heisenberg-Weyl algebras that have been applied extensively in the past (see Refs. [1,2] and references therein), the so-called q-oscillator commutation relation [3]

$$a\ a^{\dagger} - q\ a^{\dagger}\ a = 1, \quad q \in \mathrm{R}, \tag{1}$$

has deserved special attention. It has been used, for example, in the case $0<q<1$ to describe particles with small violations of Fermi and Bose statistics (quons) [4]. The appearance of quantum groups [5,6] has increased the interest in such kind of q-commutation rules, and the Biedenharn-Macfarlane q-oscillator [7,8]

$$A\ A^{\dagger} - q^{1/2}\ A^{\dagger}\ A = q^{-N/2}, \tag{2}$$

where the number operator N fulfills

$$[N,A] = -A, \quad [N,A^{\dagger}] = A^{\dagger}, \tag{3}$$

was soon introduced in order to obtain q-generalizations [9] of the usual boson mapping techniques [10]. Since then, such q-oscillator and its many-body generalizations [11,12] have been widely used to construct new effective anharmonic Hamiltonians related, for instance, with nonclassical states of light [13,14], vibrational spectra [15], and superfluidity [16].

Although the full representation theory of algebras (1) and (18) has been developed [2,17] (in fact, for q a real number, both algebras are identical under the transformation $a = q^{N/4} A$), the problem of finding the associated self-adjoint q-analogs of position and momentum operators does not have a unique solution. In Ref. [18], the position operator for the q-oscillator algebra was proposed to be

$$X = e^{z\ N}(a + a^{\dagger})e^{-z\ N} \tag{4}$$

with z being a real parameter [note the formal analogy of Eq. (4) with the S_x operator which has been recently used in Ref. [19] to describe effective fermion-boson interactions through the $\mathrm{su}_q(2)$ algebra].

In this paper we consider the Lorek-Ruffing-Wess (LRW) q-oscillator [20] which has $q=e^h>1$ and fulfills

$$a\ a^{\dagger} - q^{-2M}\ a^{\dagger}\ a = 1, M = 0, 1, 2, \ldots . \tag{5}$$

The LRW q-oscillator Hamiltonian will be defined as [20]

$$H = \omega\ a^{\dagger}\ a \tag{6}$$

and a couple of Hermitian q-position X and q-momentum P operators can be naturally associated with the creation and annihilation operators (5). We stress that, since the operators X and P have a discrete spectrum, the associated q-phase space has a lattice structure [21]. Moreover, a so-called q-deformed quantum mechanics [21–27] has been developed by considering the q-Schrödinger equation coming from the Hamiltonian

$$H = \frac{1}{2}P^2 + V(X), \tag{7}$$

which is defined in terms of the above mentioned q-position and momentum operators.

Following the work of Ref. [20], we have considered the canonical realization of Hamiltonian (6) in terms of the usual position and momentum operators x and p. Hamiltonian (6) is expanded as a power series of the parameter h. To leading order in $\sqrt{hM}$, the deformed Hamiltonian (6) turns out to be just the usual quantum mechanical oscillator plus some interaction terms which depend linearly on the momentum p. These p-dependent terms are responsible for the presence of

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local interactions, but no lattice structure is now needed for the phase space, which remains the usual continuous one. Moreover, the spectrum of this h-harmonic oscillator shows, at certain energy, a transition to a confined, finite range, regime. In other words, a certain portion of the spectrum of an h-deformed harmonic oscillator may possibly be interpreted as the spectrum of a finite range potential. We recall that p-dependent terms have been previously derived in the context of the so-called nonstandard q-deformed algebras [28], which give rise in a natural way to different types of non-Hermitian Hamiltonians, a class of objects that could be interesting in various fields of physics, among them, e.g., the physics of resonances [29] and Poeschl-Teller symmetric quantum mechanics [30].

In this work we aim at establishing a link between the h-deformation (or "truncated" q-deformation) and the finite range structure of the resulting Hamiltonian, that is, to relate the presence of p-dependent terms in the h-deformed harmonic oscillator with the appearance of finite range interactions. We shall show the existence of such a correspondence, for the case of a one-dimensional harmonic oscillator. There, the effect of the q-deformation materializes in a certain region of the spectrum, which looks like the one belonging to a one-dimensional finite range potential. In other words, we conjecture that the resulting effect of the q-deformation, and the q-deformation itself, may be related to the boundary conditions associated with the problem. In order to support our conjecture, we shall compare the resulting spectrum with the one corresponding to a central, finite range, volume-type Woods-Saxon potential. In this manner we aim at fixing realistic values of q, which is a nonobservable quantity, from observables, such as the depth of the potential and its range.

The needed background, about the LRW q-deformed harmonic oscillator, is presented in Sec. II, together with the expansion of the Hamiltonian. Section III is devoted to the discussion of the numerical results. There, we show and discuss the result of the calculations of the spectrum and wave functions of the q-deformed harmonic oscillator with the addition of the momentum dependent interactions. Conclusions are drawn in Sec. IV.

II. FORMALISM

A. The Lorek-Ruffing-Wess *q*-oscillator

Hereafter we shall follow the notation of Ref. [20]. If q denotes the deformation parameter, such that $q=e^h$, with $q>1$, and U is a unitary matrix, the relationships (deformed commutation rules) of the LRW q-Heisenberg-Weyl algebra are

$$\sqrt{q}X\,P-\frac{1}{\sqrt{q}}P\,X=iU,$$

$$U\,X-\frac{1}{q}X\,U=0,$$

$$U\,P-qP\,U=0. \tag{8}$$

The q-position X and momentum P operators defined in Eq. (8) are Hermitian. We can now express the creation and annihilation operators fulfilling the LRW algebra (5) as

$$a=\alpha\,U^{-2M}+\beta\,U^{-M}P, \tag{9}$$

$$a^{\dagger}=\bar{\alpha}\,U^{2M}+\bar{\beta}\,U^{M}P, \tag{10}$$

where $M=0,1,2,\ldots$, and α and β are complex amplitudes. With these expressions in mind, the LRW q-oscillator Hamiltonian is defined as in Eq. (6).

B. Leading order interactions

We consider the following realization [20] of the operators P and U:

$$P=\hat{p},$$

$$U=q^{-\frac{i}{2}(\hat{x}\hat{p}+\hat{p}\hat{x})},$$

where

$$\hat{x}=x,$$

$$\hat{p}=p+\frac{1}{\sqrt{1-q^{-2M}}}\gamma. \tag{11}$$

In these expressions x and $p=-i\,\partial/\partial x$ are the ordinary canonical conjugate variables and $\gamma=\pm\sqrt{2m\omega}$. In this way, the annihilation and creation operators a, Eq. (9), and $a^{\dagger}$, Eq. (10), can be written as power series of the parameter $\sqrt{hM}$. This is achieved by expanding the exponential appearing in U. The result is

$$a=\sum_{k=0}^{\infty}\sum_{l=0}^{k}c_{kl}(Mh)^k\hat{F}_{kl},$$

$$c_{kl}=\frac{(-1)^l(i)^{k+l}}{2^l l\,!\,(k-l)\,!},$$

$$\hat{F}_{kl}=\beta(\hat{x}\hat{p})^{k-l}p+(2^k\alpha+\beta\zeta)(\hat{x}\hat{p})^{k-l}, \tag{12}$$

with $\zeta=1/\sqrt{(1-q^{-2M})}\gamma$. A similar expression holds for $a^{\dagger}$. With these expressions we are now in a condition to rewrite the Hamiltonian. To leading order in $\sqrt{hM}$ one gets

$$H=H_0+\frac{1}{4}\sqrt{\frac{hM}{m\omega}}\omega[2p-3m\omega(x^2p+p\,x^2)], \tag{13}$$

where H_0 is the undeformed harmonic oscillator,

$$H_0=\frac{p^2}{2m}+\frac{m\omega^2x^2}{2}. \tag{14}$$

Hamiltonian (13) can also be written as

$$H = H_0 + \frac{\sqrt{hM}}{4}\omega\left[2b_0 p + \frac{6i}{b_0}x - \frac{6}{b_0}x^2 p\right], \tag{15}$$

where b_0 is the oscillator length ($b_0 = \sqrt{1/m\omega}$) in units of $\hbar = 1$.

The leading order Hamiltonian of Eq. (15) is, basically, a quartic anharmonic oscillator. This can be shown by performing on Eq. (15) the gauge transformation:

$$\Phi(x) = \phi(x)\exp[iA(x)], \tag{16}$$

where

$$A(x) = \frac{\sqrt{hM}\; x\;(-b_0^2 + x^2)}{2b_0^3}. \tag{17}$$

The resulting differential equation is

$$-\frac{\phi''}{2m} + \frac{(1+24g^2)x^2\omega\phi(x)}{2b_0^2} - \frac{18g^2x^4\omega\phi(x)}{b_0^4} - 2g^2\omega\phi(x) = E\phi(x). \tag{18}$$

The above expression is similar, for small values of x, to the Poeschl-Teller potential

$$V(x) = \frac{V_0}{\cosh^2(x/R)} = V_0 - V_0\frac{x^2}{R^2} + V_0\frac{2x^4}{3R^4} + \cdots. \tag{19}$$

Thus, from a formal point of view we can argue that q-deformations are indeed similar to finite range potentials, as shown above.

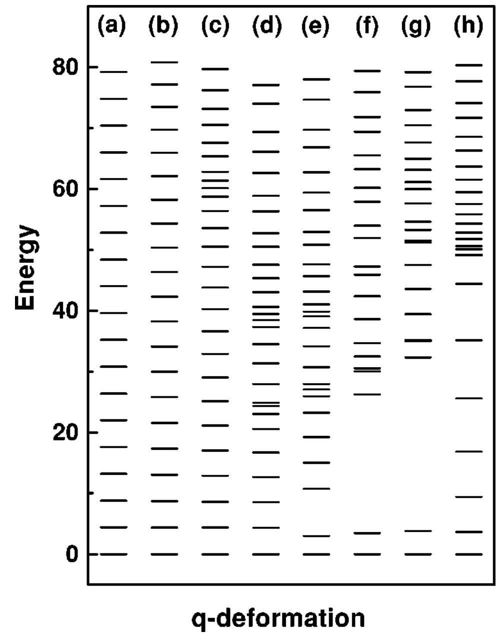

FIG. 1. Eigenvalues of the harmonic oscillator, case (a); of the q-deformed harmonic oscillator, cases (b)–(g); and of the Woods-Saxon potential, case (h). The values are shown in arbitrary units. The parameters used in the calculations are, respectively, $\hbar\omega = 4.4$ [cases (a)–(g)]; and $g = 0.001$ [case (b)], $g = 0.015$ [case (c)], $g = 0.019$ [case (d)], $g = 0.022$ [case (e)], $g = 0.024$ [case (f)], and $g = 0.025$ [case (g)]. Case (h) corresponds to the Woods-Saxon potential with $V_0 = -50$, $a = 0.5$, and $R_0 = 1.1$. The calculations have been performed in a basis with 65 harmonic oscillator shells.

III. RESULTS AND DISCUSSION

We have calculated the eigenvalues of H, of Eq. (13), for various values of the coupling $g = \sqrt{hM}/4$. The results are shown in Fig. 1. There, for small values of g, cases (b)–(d), the low-energy portion of the spectrum remains nearly invariant, with respect to the pure harmonic oscillator, while for larger values of g, cases (e)–(g), the high-energy part of the spectrum becomes more dense and it looks similar, case (g), to the spectrum of the finite range Woods-Saxon potential,

$$V(r) = \frac{V_0}{1 + e^{(r-R_0)/a_0}}, \tag{20}$$

case (h).

Note that a direct comparison of cases (a) and (h), the pure harmonic oscillator and the Woods-Saxon potential, respectively, may be performed for some low-lying states only, but both potentials look very different at high energies, due to the unbounded character of the harmonic oscillator. This is a direct consequence of the boundary conditions which are imposed on each case. From these, we may conclude that the spectrum of the q-deformed harmonic oscillator may approach the spectrum of the finite range potential, as shown by cases (b)–(g). This is a definite effect of the q-deformation. The use of relatively large values of g causes the appearance of a gap at low energy, as observed in cases (f) and (g). This effect may be explained by the truncation performed at the level of the Hamiltonian.

In order to explore the effects of the q-deformation we have calculated the spectra of q-deformed harmonic oscillator of different frequencies and deformations, with a standard Woods-Saxon potential. The result is shown in Fig. 2. As indicated by the captions of this figure, the dimension of the basis used to diagonalize the Hamiltonians of Eqs. (13) and (20), for each of the shown cases, is considerably large. This is to ensure that truncation is not affecting the results. Since the agreement between the positive energy parts of the spectra persists, the effect may be only attributed to the q-deformation. The states of the q-deformed harmonic oscillator have been shifted, so that its lowest state coincides with the lowest eigenvalue of the finite range potential. From the result shown in Fig. 2 one can see that both spectral distributions nearly coincide over the full range of energies.

The similarities between the results of the q-deformed harmonic oscillator and the ones obtained with a finite range potential, such as the Woods-Saxon potential of Eq. (20), are better illustrated, actually, by comparing the shapes of the potentials introduced in Sec. II. Figure 3 shows the dependence of the potentials, with the coordinate, in the region of interest. The curves have been calculated at leading order and for relatively small values of x. For values of x larger than 1.4 the three curves change their curvatures, as happens

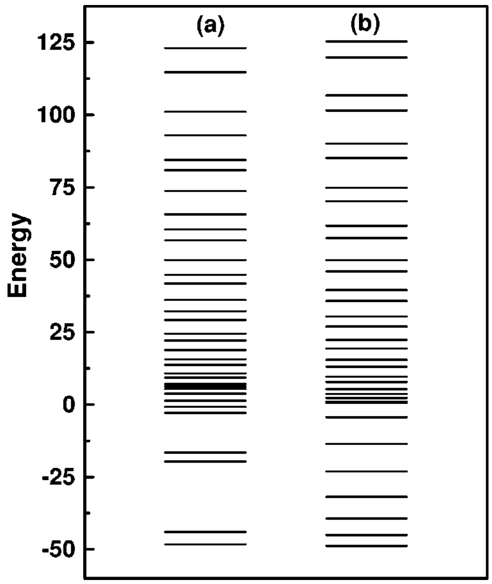

FIG. 2. Eigenvalues (in arbitrary units) for a q-deformed harmonic oscillator with $\hbar\omega$=2.8 and g=0.044 [case (a)] and of the Woods-Saxon potential, case (b), with the parameters given in the caption of Fig. 1. The calculations have been performed in a basis consisting of 45 harmonic oscillator shells.

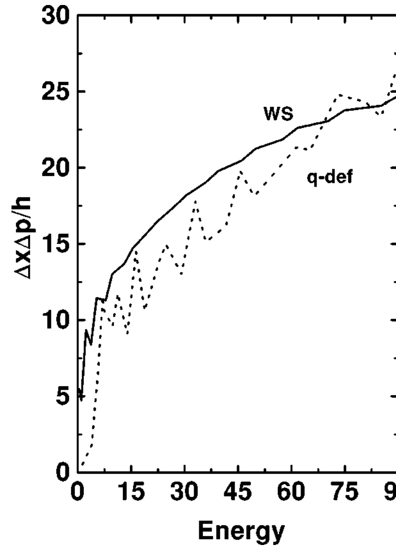

FIG. 4. Uncertainties $\Delta x \Delta p$, divided by $\hbar$, calculated by using the wave functions obtained from the diagonalization of the Woods-Saxon (WS, solid-line) and q-deformed harmonic oscillator (q-def, dotted-line), for the positive-energy eigenvalues (denoted as energy-values in arbitrary units). The parameters of the potentials are given in the caption of Fig. 2.

in the case of the q-deformed oscillator (q-osc) near x=1.4. The parameters used in the calculations are given in the caption of Fig. 3.

Concerning the structure of the wave functions, the uncertainties $\Delta x \Delta p$ for each eigenvalue of the finite range potential

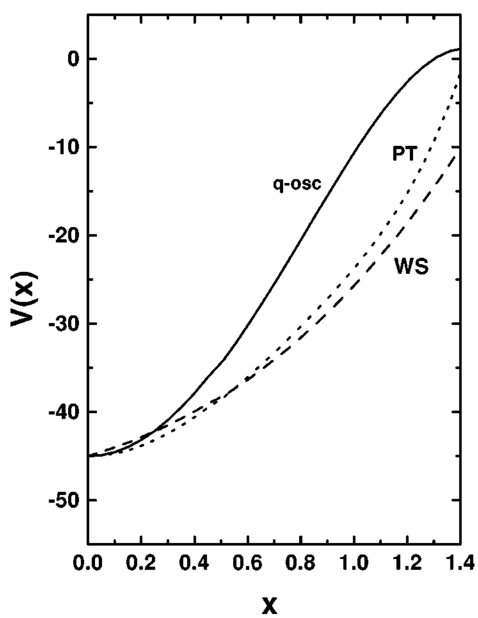

FIG. 3. The curves show the results for the potential $V(x)$ corresponding to the q-deformed harmonic oscillator of Eq. (18) (q-osc, solid line), Poeschl-Teller (PT, dotted-line), and Woods-Saxon (WS, dashed-line) potentials. The parameters are $\hbar\omega$=6.98, g=1.8 (q-osc); V_0=−50, R_0=1.1, a_0=0.5 (WS), and V_0=−45, R_0=1.26 (PT). The values of $V(x)$ and x are given in arbitrary units.

(20) and for the q-deformed harmonic oscillator (13) are comparable for the eigenvalues of positive energies. The results are shown in Fig. 4. The q-deformed harmonic oscillator exhibits, however, larger values for the product of uncertainties at low energies in the negative energy part of the spectrum of Fig. 2. Thus, it may be concluded that the wave functions of the finite range potential and those of the q-deformed oscillator produce comparable observables.

Both features, namely, (a) the similarity of the eigenvalues and density of eigenvalues, and (b) the comparable wave functions, strongly suggest the existence of a correspondence between the q-deformation and the finite range structure of a potential. Moreover, it points to a more fundamental equivalence between a q-deformed algebra, such as the one chosen to "deform" a certain Hamiltonian, and the boundary conditions which one imposes on the corresponding eigenvalue problem. Although this correspondence has to be probed yet, from a pure mathematical point of view, physical intuition signals the possible existence of a trade-off mechanism between added interactions, like the one generated by the q-deformation of a certain algebraic structure, and boundary conditions.

IV. CONCLUSIONS

In this work we have revisited the problem of the LRW q-deformed harmonic oscillator, introduced in the work of Ref. [20], by considering only the leading order correction introduced by the deformation. In this way, no discretization of the quantum phase space is needed, and from the comparison between the eigenvalue densities and deviations of ca–

nonical conjugate variables we conjecture that a correspondence between *q*-deformation effects and boundary conditions (e.g, the finite range behavior of a central potential) may exist. In the example which we have presented in this work, the correspondence is clearly illustrated for the change of the regime of the eigenvalue density of an unbounded harmonic spectrum, which behaves, after *q*-deformations, like the spectrum of a central potential of finite range.

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