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Squeezing in a spin chain with site-dependent periodic and long-range interactions

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ABSTRACT

Numerical and analytical results for the squeezing factor, ζ^2, in a pseudo-spin s-1/2 chain. The open chain is composed by N two-level atoms with site-dependent interactions. The time evolution of the squeezing factor is studied, as well as its dependence on the number of atoms and on the interactions. It is found that long-range interactions may optimize the degree of spin squeezing.

1. Introduction

Recent developments in quantum computation and quantum optics have renewed the interest in the study of spin systems and their interactions [1,2]. In particular, the problem of the spin reconstruction in quantum non-demolition measurements [3] is closely related to spin-squeezing observables [4]. However, there are some questions about the modelling and physical properties of spin systems which must be answered in order to support the claim that spin chains are suitable devices to transmit quantum information. Among them we have selected, for this work, the question about the persistency of spin orientation in presence of spin–spin interactions [5–9]. One may then investigate, as a convenient tool, the spin response of a Heisenberg spin-chain with different interactions [10–13]. Based on our previous experience [14–16] we shall consider anisotropic Heisenberg spin-chains, because the anisotropy of the spin-chain interactions do play a role in building the balance between the components of the total spin. To accomplish this goal we have selected, from the literature, periodic, Gaussian and periodic long-range spin–spin interactions [10–12].

In the present work we calculate spin-squeezing observables [17], in a system of interacting two-level atoms distributed in the sites of an open chain. The calculation of spin expectation values proceeds *via* the use of a density matrix formalism developed previously [14].

The Letter is organized as follows. The formalism is described in Section 2, results and discussions are presented in Section 3. Conclusions are drawn in Section 4.

2. Formalism

The frame of reference, for the formalism which is presented below, is the treatment of spin-1/2 anti-ferromagnetic (or ferromagnetic) chains with site–site interactions [10,11]. Formal aspects of the solution are well known [12] and we shall omit them here, for the sake of brevity. Essentially, one must define the Hamiltonian of the spin-chain, the basis, and, by diagonalization, obtain the exact solution for a reasonable number of atoms [16]. In the next subsections we shall focus on specific details of the solution for different interactions. Concerning the time evolution of the observables, we use the density-matrix-formalism which we have developed in [14]. For the spin-squeezing factor, we have adopted the definition of Refs. [8,9].

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2.1. Asymmetric spin-chains

In the following we shall discuss few examples of spin systems which are exactly solvable by diagonalization. The spectrum of each atom consists of two levels of spin $s=\frac{1}{2}$. We refer to the system as a spin-$\frac{1}{2}$ chain, for simplicity. However, since we are talking about two-level atoms, the term pseudo-spin-$\frac{1}{2}$-chain may be more appropriate. The atoms interact by their spins, and the spin–spin interaction is decomposed in spin components with, in general, different strengths. We write the Hamiltonian of the system as

$$H=\sum_{i\neq j=1}^{N} g(i,j)(\lambda_x s_{x,i}s_{x,j}+\lambda_y s_{y,i}s_{y,j}+\lambda_z s_{z,i}s_{z,j}). \tag{1}$$

The interaction (1) is a generalization of the interaction between spin sites $\bar{S}_i\cdot\bar{S}_j$ weighted by directional couplings

$$g_\gamma(i,j)=\lambda_\gamma g(i,j),\quad \gamma=x,y,z,\ i=1,2,\ldots,N. \tag{2}$$

The spin raising and lowering operators $s_{\pm,i}=s_{x,i}\pm is_{y,i}$, and the spin-$z$ operator $s_{z,i}$, obey $su(2)$-commutation rules, and the Hamiltonian of Eq. (1) may be written as

$$\begin{aligned}H=\sum_{i\neq j=1}^{N} g(i,j)\big(&\lambda_-(s_{+,i}s_{+,j}+s_{-,i}s_{-,j})\\&+\lambda_+(s_{+,i}s_{-,j}+s_{-,i}s_{+,j})+\lambda_z s_{z,i}s_{z,j}\big),\end{aligned} \tag{3}$$

with $\lambda_\pm=\frac{1}{4}(\lambda_x\pm\lambda_y)$. The product states

$$|m\rangle=|k_1,k_2,\ldots,k_i,\ldots,k_N\rangle=\prod_{i=1}^{N}s_{+,i}^{k_i}|0\rangle,\quad k_i=0,1, \tag{4}$$

define a basis where the Hamiltonian (3) can be diagonalized; the index k_i labels the state of the i-th atom, and the values $k_i=0$ and 1 are associated with the ground state ($s_z=-1/2$) and first excited state ($s_z=1/2$) of each atom, respectively. The state $|0\rangle$ is the vacuum, and by definition in $|0\rangle$ all atoms are in the lowest ($s_z=-1/2$) atomic level. The matrix elements of H (3) in the basis (4) are given by the expression

$$\begin{aligned}\langle m'|H|m\rangle=\sum_{i\neq j} g(i,j)\big(&\lambda_-\delta(k_i+1,k_i')\delta(k_j+1,k_j')\\&+\lambda_-\delta(k_i-1,k_i')\delta(k_j-1,k_j')\\&+\lambda_+\delta(k_i+1,k_i')\delta(k_j-1,k_j')\\&+\lambda_+\delta(k_i-1,k_i')\delta(k_j+1,k_j')\\&+\lambda_z(k_i-1/2)(k_j-1/2)\delta(k_i,k_i')\delta(k_j,k_j')\big).\end{aligned} \tag{5}$$

The non-diagonal elements of the matrix (5) connect states with $\Delta k=0$ and ±2, where $k=\sum_i k_i-N/2$ is the projection of the total spin in the z direction. In particular, for the choice $\lambda_x=\lambda_y=\lambda_\perp$, the operator $S_z=\sum_{i=1}^{N}s_{z,i}$ commutes with the Hamiltonian (3), and the matrix elements (5) are written

$$\begin{aligned}\langle m'|H|m\rangle=\sum_{i\neq j} g(i,j)\bigg(&\frac{\lambda_\perp}{2}\big(\delta(k_i+1,k_i')\delta(k_j-1,k_j')\\&+\delta(k_i-1,k_i')\delta(k_j+1,k_j')\big)\\&+\lambda_z\delta(k_i,k_i')\delta(k_j,k_j')\Big(k_i-\frac{1}{2}\Big)\Big(k_j-\frac{1}{2}\Big)\bigg).\end{aligned} \tag{6}$$

These are the elements needed to calculate, exactly, the eigenvalues and eigenvectors, and with them the corresponding density matrix [14].

In the basis of eigenvectors of H, constructed as discussed before, the time evolution of a given operator O is expressed as

$$O(t)=U^\dagger(t)OU(t),\quad U(t)=e^{-iHt/\hbar}. \tag{7}$$

The expectation value $\langle O(t)\rangle$ is then written

$$\begin{aligned}\big\langle O(t)\big\rangle&=\mathrm{Tr}\big(\rho(t)O\big)\\&=\sum_{\alpha,\beta}\langle\beta|I\rangle\langle I|\alpha\rangle\langle\alpha|O|\beta\rangle e^{-i(E_\alpha-E_\beta)t/\hbar},\end{aligned} \tag{8}$$

where we have defined the density operator $\rho(t)=U^\dagger(t)\rho(0)U(t)$, being $\rho(0)=|I\rangle\langle I|$; the state $|I\rangle$ is the initial state of the system, while $\{E_\alpha\}$ and $\{|\alpha\rangle\}$ are the α-th eigenvalue and eigenvector of the Hamiltonian.

The expression (8) can be written in a more compact form in terms of the overlap of the initial state $|I\rangle$ with the eigenvectors $\{|\alpha\rangle\}$, that is [14]

$$\begin{aligned}\big\langle O(t)\big\rangle&=\sum_{n,m}T^*(n)\langle n|O|m\rangle T(m),\\T(m)&=\sum_{\alpha n}c_{\alpha n}^*c_{\alpha m}\langle n|I\rangle e^{iE_\alpha t/\hbar}.\end{aligned} \tag{9}$$

In the above equation $|n\rangle$ is an element of the basis (4), and the coefficient $c_{\alpha n}$ is the amplitude of $|n\rangle$ in the eigenstate $|\alpha\rangle$ of the Hamiltonian of Eq. (3).

2.2. The squeezing factor

We have adopted, for the squeezing factor, the definition [8]

$$\zeta^2=\frac{N(\Delta S_n)^2}{|\langle S\rangle|^2}, \tag{10}$$

where N is the number of atoms in the chain, and S_n is the component of the total spin in a direction perpendicular to $\langle S\rangle$, that is the direction defined by the unitary vector

$$\begin{aligned}&\breve{n}=(\sin\theta_n\cos\phi_n,\sin\theta_n\sin\phi_n,\cos\theta_n),\\&\breve{n}\cdot\langle\vec{S}\rangle=0.\end{aligned} \tag{11}$$

2.3. The initial state

In all of the present calculations we have considered the state

$$\begin{aligned}&|I\rangle=\big(1+|z|^2\big)^{-N/2}\sum_{m=0}^{N}\frac{z^m}{m!}S_+^m|0\rangle,\\&z=e^{-i(\phi_0-\pi)}\tan(\theta_0/2),\end{aligned} \tag{12}$$

as the initial condition. This state is not eigenstate of H, and it is defined by the orientation angles ϕ_0 and θ_0 [17]. The operator S_+ is the total spin-raising operator $S_+=\sum_{i=1}^{N}s_{+,i}$.

2.4. Some analytically solvable cases

We shall begin with the case with $N=2$ atoms, and consider different interactions between the sites of the chain, which is an open chain as determined by the definition of the above introduced Hamiltonian.

2.4.1. Harmonic interaction

We write the interaction between atoms with $g(i,j) = \sin(|i-j|\frac{\pi}{2})$ and $\lambda_z = 2\lambda_\perp$. This choice of the couplings may be arbitrary, but it is supported by the fact that unless the symmetry of the interaction (1) is broken, there will not be a net gain in the squeezing (10), as we shall discuss in the subsection below. The diagonalization of the Hamiltonian yields the eigenvalues ϵ_α and eigenvectors Ψ_α (with $\alpha = 1,2,3,4$)

$$\begin{aligned}
&\epsilon_1 = \lambda_\perp, \qquad |\Psi_1\rangle = |0,0\rangle,\\
&\epsilon_2 = 0, \qquad |\Psi_2\rangle = \frac{1}{\sqrt{2}}\big(|1,0\rangle + |0,1\rangle\big),\\
&\epsilon_3 = -2\lambda_\perp, \qquad |\Psi_3\rangle = \frac{1}{\sqrt{2}}\big(|1,0\rangle - |0,1\rangle\big),\\
&\epsilon_4 = \lambda_\perp, \qquad |\Psi_4\rangle = |1,1\rangle.
\end{aligned} \tag{13}$$

With these eigenvalues and eigenvectors we can calculate the time evolution of the spin observables, starting from the initial condition (12). One obtains

$$\begin{aligned}
&\big\langle S_z(t)\big\rangle = -\cos\theta_0,\\
&\big\langle S_z^2(t)\big\rangle = \frac{1}{2} + \frac{1}{2}\cos^2\theta_0.
\end{aligned} \tag{14}$$

This calculation can be repeated for different orientations of the initial condition, like: $\theta_0 = \frac{\pi}{2}$ and $\phi_0 = 0$. For this choice the expectation value of the z-component of the total spin vanishes, $\langle S_z\rangle = 0$, and

$$\begin{aligned}
&\langle\vec{S}\rangle = \big(\langle S_x\rangle, 0, 0\big),\\
&\langle S_x\rangle = -\cos\left(\frac{\lambda_\perp t}{\hbar}\right).
\end{aligned} \tag{15}$$

The unitary vector which defines the direction normal to $\langle\vec{S}\rangle$ is $\check{n} = (0, \pm\sin\theta_n, \cos\theta_n)$. The positive sign corresponds to the choice $\phi_n = \frac{\pi}{2}$ and the minus sign to $\phi_n = \frac{3\pi}{2}$. With these elements, the formal expression for the squeezing factor ζ^2 reads

$$\zeta^2 = \frac{1 \pm 2\sin\theta_n\cos\theta_n\sin(\frac{\lambda_\perp t}{\hbar})}{\cos^2(\frac{\lambda_\perp t}{\hbar})}. \tag{16}$$

Analogously, the choice $\theta_0 = 0$ and $\phi_0 = 0$ yields $\zeta^2 = 1$ (no squeezing). The case with $\theta_0 = \frac{\pi}{2}$, $\phi_0 = \frac{\pi}{2}$ is similar to case $\theta_0 = \frac{\pi}{2}$, $\phi_0 = 0$, and it yields the same expression for ζ^2 (16).

2.4.2. Gaussian interaction

The interaction between spin sites may be modelled by a Gaussian form-factor

$$g(i,j) = e^{-\frac{1}{2}(i-j)^2}. \tag{17}$$

The solutions of the eigenvalue problem, for this interaction, are written

$$\begin{aligned}
&\epsilon_1 = \frac{\lambda_z}{2}e^{-1/2}, \qquad |\Psi_1\rangle = |0,0\rangle,\\
&\epsilon_2 = \left(\lambda_\perp - \frac{\lambda_z}{2}\right)e^{-1/2}, \qquad |\Psi_2\rangle = \frac{1}{\sqrt{2}}\big(|1,0\rangle + |0,1\rangle\big),\\
&\epsilon_3 = -\left(\lambda_\perp + \frac{\lambda_z}{2}\right)e^{-1/2}, \qquad |\Psi_3\rangle = \frac{1}{\sqrt{2}}\big(|1,0\rangle - |0,1\rangle\big),\\
&\epsilon_4 = \frac{\lambda_z}{2}e^{-1/2}, \qquad |\Psi_4\rangle = |1,1\rangle.
\end{aligned} \tag{18}$$

With the choice $\lambda_z = 2\lambda_\perp$ the above expressions reduce to the solutions (13), but for the factor $e^{-1/2}$, and for the orientation angles $\theta_0 = \frac{\pi}{2}$, $\phi_0 = 0$ one obtains

$$\zeta^2 = \frac{1 \pm 2\sin\theta_n\cos\theta_n\sin(\frac{\lambda_\perp t}{\hbar}e^{-1/2})}{\cos^2(\frac{\lambda_\perp t}{\hbar}e^{-1/2})}, \tag{19}$$

and the same expression holds for $\theta_0 = \frac{\pi}{2}$, $\phi_0 = \frac{\pi}{2}$. A similar analysis may be performed for the case with $N = 3$ atoms. The solution, for both interactions, is also analytical, and it leads to results which are similar to Eqs. (16) and (19) (see the discussion presented in the following subsection).

2.4.3. The case of $N = 3$ atoms with non-symmetric interactions

In this subsection we shall discuss, by solving analytically the case of three atoms, the effect of the asymmetry of the interaction upon the spin-squeezing factor ζ^2. The main notion behind this is the symmetry-breaking mechanism induced by the couplings λ_i of Eq. (3). In this example, the diagonalization of (3) is performed in the basis of eight states which results from the enumeration of the three sites (atoms) of the chain and two states (spin states in each site). The eigenvalues of the corresponding secular equation are

$$\begin{aligned}
&\epsilon_1 = \epsilon_8 = \lambda_z/2,\\
&\epsilon_2 = \epsilon_5 = 0,\\
&\epsilon_3 = \epsilon_6 = -\lambda_z/4 - \Delta,\\
&\epsilon_4 = \epsilon_7 = -\lambda_z/4 + \Delta
\end{aligned} \tag{20}$$

with

$$\Delta = \frac{\sqrt{\lambda_z^2 + 8\lambda_\perp^2}}{4}.$$

The time dependent expectation value of the spin components can be calculated explicitly, and their expressions read

$$\begin{aligned}
&\langle S_z\rangle = -\frac{3}{2}c_0,\\
&\big\langle S_z^2\big\rangle = \frac{3}{4} + \frac{3}{2}c_0^2,\\
&\big\langle S_+^2\big\rangle = e^{-2i(\phi_0-\pi)}\frac{1}{2}s_0^2(A - iBc_0),\\
&\langle S_+\rangle = e^{-i(\phi_0-\pi)}\frac{1}{2}s_0\left[\frac{1}{2}s_0^2C + \left(1 - \frac{1}{2}s_0^2\right)A - iBc_0\right],\\
&\big\langle\{S_+, S_-\}\big\rangle = 3 + s_0^2C,\\
&\big\langle\{S_+, S_z\}\big\rangle = e^{-i(\phi_0-\pi)}s_0\left[-c_0A + i\left(1 - \frac{1}{2}s_0^2\right)B\right],
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\langle S_x\rangle = \mathrm{Re}\langle S_+\rangle,\\
&\langle S_y\rangle = \mathrm{Im}\langle S_+\rangle,\\
&\big\langle S_x^2\big\rangle = \frac{1}{2}\,\mathrm{Re}\big\langle S_+^2\big\rangle + \frac{1}{4}\big\langle\{S_+, S_-\}\big\rangle,\\
&\big\langle S_y^2\big\rangle = -\frac{1}{2}\,\mathrm{Re}\big\langle S_+^2\big\rangle + \frac{1}{4}\big\langle\{S_+, S_-\}\big\rangle
\end{aligned} \tag{22}$$

with

$$\begin{aligned}
&c_0 = \cos(\theta_0),\\
&s_0 = \sin(\theta_0),
\end{aligned}$$

in correspondence with the definition of the orientation angles of the initial state (12). The other quantities entering the definition of the spin expectation values are

$$A = \left[\frac{3}{2} + \frac{1}{2}(\alpha - 2\beta)\right]\cos\left(\Delta + \frac{3}{4}\lambda_z t\right) + \left[\frac{3}{2} - \frac{1}{2}(\alpha - 2\beta)\right]\cos\left(\Delta - \frac{3}{4}\lambda_z\right)t,$$

$$B = \left[\frac{3}{2} + \frac{1}{2}(\alpha - 2\beta)\right]\sin\left(\Delta + \frac{3}{4}\lambda_z\right)t - \left[\frac{3}{2} - \frac{1}{2}(\alpha - 2\beta)\right]\sin\left(\Delta - \frac{3}{4}\lambda_z\right)t,$$

$$C = 3 - (\alpha + \beta)^2 \sin^2(\Delta t) \tag{23}$$

with

$$\alpha = \frac{\lambda_z - 4\lambda_\perp}{4\Delta},$$

$$\beta = \frac{\lambda_z + 2\lambda_\perp}{4\Delta},$$

$$3 = \alpha^2 + 2\beta^2. \tag{24}$$

It is then straightforward to show that for a symmetric interaction ($\lambda_z = \lambda_\perp$) one obtains

$$\Delta = \frac{3}{4}\lambda_z,$$

$$\alpha = -1,$$

$$\beta = 1,$$

$$A = 3,$$

$$B = 0,$$

$$C = 3. \tag{25}$$

With these results, the expectation values (21) are written

$$\langle S_z\rangle = -\frac{3}{2}c_0,$$

$$\langle S_z^2\rangle = \frac{3}{4} + \frac{3}{2}c_0^2,$$

$$\langle S_+^2\rangle = e^{-2i(\phi_0 - \pi)}\frac{3}{2}s_0^2,$$

$$\langle S_+\rangle = e^{-i(\phi_0 - \pi)}\frac{3}{2}s_0,$$

$$\langle\{S_+, S_-\}\rangle = 3\left(1 + s_0^2\right),$$

$$\langle\{S_+, S_z\}\rangle = -3e^{-i(\phi_0 - \pi)}s_0 c_0. \tag{26}$$

The direction of the total spin is determined by the ratios

$$\frac{\langle S_x\rangle}{|\langle \mathbf{S}\rangle|} = -s_0\cos(\phi_0),$$

$$\frac{\langle S_y\rangle}{|\langle \mathbf{S}\rangle|} = s_0\sin(\phi_0),$$

$$\frac{\langle S_z\rangle}{|\langle \mathbf{S}\rangle|} = -c_0. \tag{27}$$

Therefore, from these relations, one obtains

$$(\Delta S_n)^2 = \frac{3}{4},$$

$$\left|\langle S\rangle\right| = \frac{3}{2}, \tag{28}$$

and, consequently[1]

$$\zeta^2 = N\frac{(\Delta S_n)^2}{|\langle S\rangle|^2} = 1. \tag{29}$$

[1] These results hold independently of the value of the angle ϕ_0.

It means that, in order to obtain values of $\zeta^2 < 1$, the symmetry of the Hamiltonian must be broken (see also [14–16]). We shall make use of Eqs. (16) and (19) in order to gain some insight about the results for cases which cannot be solved analytically.

2.4.4. The TDA framework for a large number of atoms

The Tamm–Dancoff Approximation (TDA) may be used to calculate spin observables in the limit of a large number of atoms. The TDA is based on the boson expansion of the spin operators and upon a subsequent mapping onto the one-phonon basis. This is done by applying the following steps:

(a) Boson expansion of the spin operators: The spin operators are expressed in terms of boson operators associated to each site of the chain, to read [18]

$$S_+ = \sum_i b_i^+,$$

$$S_- = \sum_i b_i,$$

$$S_z = \sum_i b_i^+ b_i - \frac{N}{2}. \tag{30}$$

The vacuum state $|0\rangle$ is defined as the state with all atoms in their lower level, such that for bosons which lift the spin of the i-th site, it follows that $b_i|0\rangle = 0$.[2] Clearly, the operators (30) obey a SU(2) algebra.

(b) One-phonon basis: The boson operators b_i^+ (the subindex i indicates the site) can be transformed to a new basis of boson operators Γ_n by the linear combinations

$$\Gamma_n^+ = \sum_i X_n(i) b_i^+. \tag{31}$$

The new operators are acting on the same vacuum $|0\rangle$, thus

$$\Gamma_n|0\rangle = 0 \tag{32}$$

and the set of states generated by them are written

$$|n\rangle = \Gamma_n^+|0\rangle. \tag{33}$$

Eq. (31) is inverted, leading to the inverse transformation

$$b_i^+ = \sum_n X_n^*(i)\Gamma_n^+ \tag{34}$$

since the transformation (31) preserves the commutation relation of the original boson operators.

(c) TDA diagonalization: The Hamiltonian of Eq. (3) is transformed to the boson basis of the b_i^+ (and b_i) operators and ultimately to the basis of the one-phonon operators (31). The TDA image of the Hamiltonian reads

$$H_{\mathrm{TDA}} = \sum_i \epsilon_i\left(\hat{n}_i - \frac{1}{4}\right) + \sum_{i\neq j}\lambda_\perp g_{ij} b_i^+ b_j = \sum_n E_n \Gamma_n^+\Gamma_n \tag{35}$$

with

$$\epsilon_i = \sum_{j\neq i}\lambda_z g_{ij}. \tag{36}$$

[2] If one represents atomic levels of a given site by the action of creation and annihilation operators $c_{i,\pm}^+$ and $c_{i,\pm}$, the boson creation operator is defined as $b_i^+ = c_{i,+}^+ c_{i,-}$ and the vacuum state is of the form $|0\rangle = \Pi_j c_{j,-}^+|\rangle$, such that $b_i|0\rangle = c_{i,-}^+ c_{i,+}\Pi_j c_{j,-}^+|\rangle = 0$.

In the spirit of the TDA approximation [18] the Hamiltonian is linearized by the equation of motion

$$\left[H_{\mathrm{TDA}}, \Gamma_n^+\right] = E_n \Gamma_n^+ \tag{37}$$

which leads to the secular eigenvalue equation

$$\epsilon_l X_n(l) + \sum_{k\neq l} \lambda_\perp g_{lk} X_n(k) = E_n X_n(l). \tag{38}$$

Therefore, by replacing the amplitude

$$X_n(l) = \frac{1}{E_n - \epsilon_l} \sum_{k\neq l} \lambda_\perp g_{lk} X_n(k) = \frac{\Lambda_n(l)}{E_n - \epsilon_l} \tag{39}$$

in Eq. (38) one obtains the dispersion relation for the TDA energies E_n. The TDA amplitudes are normalized

$$\sum_l \left|X_n(l)\right|^2 = 1. \tag{40}$$

(d) Initial condition: Once the TDA boson transformation is performed, the initial condition may be written in terms of the TDA phonons as

$$|I\rangle = \mathcal{N} \exp\left(z \sum_{n=1}^{N} \lambda_n \Gamma_n^\dagger\right)|0\rangle, \tag{41}$$

where $\mathcal{N}$ is the norm of the state $|I\rangle$.

(e) Spin operators in the TDA basis: As we have done before for the initial condition and the Hamiltonian, the spin operators can be written in the TDA basis:

$$S_z = \sum_n \Gamma_n^\dagger \Gamma_n - \frac{N}{2},$$
$$S_+ = \sum_n \lambda_n \Gamma_n^+ \tag{42}$$

with

$$\lambda_n = \sum_k X_n(k). \tag{43}$$

The spin squeezing factor is readily obtained after a straightforward algebraic operation which yields, at leading order, the expressions

$$\langle S_z\rangle = -\frac{N}{2} + \frac{|z|^2 N}{1+|z|^2 N},$$
$$\left\langle S_z^2\right\rangle = \frac{N^2}{4} - (N-1)\frac{N|z|^2}{1+|z|^2 N},$$
$$\langle S_+\rangle = \frac{z^*}{1+|z|^2 N} \sum_{n=1}^{N} |\lambda_n|^2 e^{i(E_n - E_0)t},$$
$$\left\langle S_+^2\right\rangle = 0,$$
$$\left\langle\{S_+, S_-\}\right\rangle = N + \frac{2|z|^2}{1+N|z|^2} \sum_n |\lambda_n|^4,$$
$$\left\langle\{S_+, S_z\}\right\rangle = -(N-1)\langle S_+\rangle, \tag{44}$$

for the spin expectation values. In the limit of large values of N one obtains

$$\langle S_z\rangle = -\frac{N}{2},$$
$$\left\langle S_z^2\right\rangle = \frac{N^2}{4},$$
$$\langle S_+\rangle = \left\langle S_+^2\right\rangle = 0,$$
$$\left\langle\{S_+, S_-\}\right\rangle = N,$$
$$\left\langle\{S_+, S_z\}\right\rangle = 0, \tag{45}$$

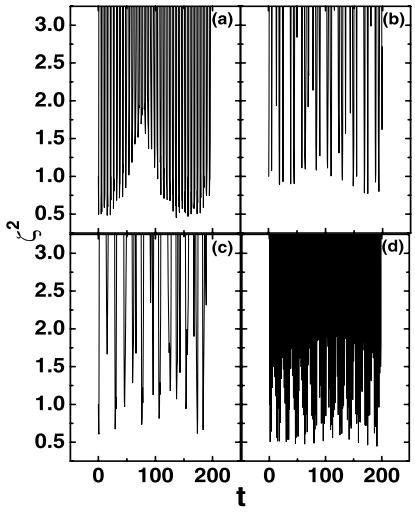

Fig. 1. Time dependence of the squeezing parameter, ζ^2, for different interactions. Insets (a)–(d) show the results corresponding to the factors $g(i, j)$ of Eq. (47). The strength $\lambda_\perp$ and λ_z are fixed at the values $\lambda_\perp = 1$ and $\lambda_z = 2$, respectively. The initial state is a coherent state (12) with $\theta_0 = \pi/2$ and $\phi_0 = 0$. The time is given in arbitrary units. The results correspond to the case with $N = 5$ atoms.

therefore, with the definition of the squeezing factor (10), it reads

$$\zeta^2{}_{\text{large } N} \to \frac{(N^2/4)|\breve{n}|^2}{(N^2/4)} = 1 \tag{46}$$

since the vector $\breve{n}$ is a unitary vector. The above results seemingly indicate that, at least in the TDA subspace, the saturation of the spin squeezing factor may be expected for a large number of atoms. Naturally, for a complete proof of the statement, an exact diagonalization for a large value of N is needed, but this task is limited for the fast growing dimensionality (2^N) as N increases.

3. Results and discussion

In this section we shall present and discuss the results of the above introduced formalism, for the squeezing factor ζ^2, for various interactions between the spin sites. In all cases we have considered two-level atoms, the spin of each level being $s = 1/2$. The factors $g(i, j)$ of the interactions, which we have diagonalized for $N \leqslant 11$, are the following

$$g(i,j) = \left|\sin\left(\frac{\pi}{2}(i-j)\right)\right|, \qquad \text{(a)}$$
$$g(i,j) = \sin\left(\frac{\pi}{2}|i-j|\right), \qquad \text{(b)}$$
$$g(i,j) = e^{-\frac{1}{2}(i-j)^2}, \qquad \text{(c)}$$
$$g(i,j) = \left(\sin\left(\frac{\pi}{N}(i-j)\right)\right)^{-2}. \qquad \text{(d)} \tag{47}$$

The associated interactions are either periodic or long-range interactions, which have been discussed in the literature, like in Refs. [10–13].

The factors $\lambda_\perp$ and λ_z of Eq. (6), are fixed at the values $\lambda_\perp = 1$ and $\lambda_z = 2$, for all cases, to enforce the asymmetry of the interactions [14–16], as explained in the previous subsections. As initial condition we have chosen the state (12) with $\theta_0 = \pi/2$, $\pi/4$ and $\pi/8$, and $\phi_0 = 0$.

In Figs. 1–3 we show the time dependence of the squeezing parameter ζ^2 for a chain with $N = 5$ atoms. The procedure consists of: (i) diagonalization of the Hamiltonian (3) in the basis (4);

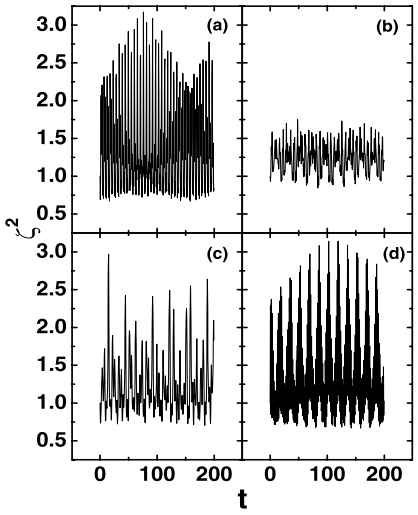

Fig. 2. The same as Fig. 1, for an initial coherent state with $\theta_0 = \pi/4$ and $\phi_0 = 0$.

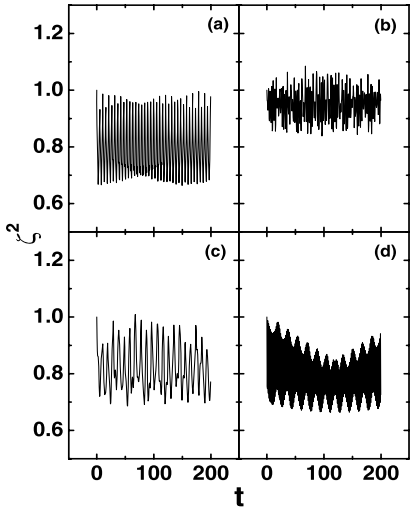

Fig. 3. The same as Fig. 1, for the initial state (12) with $\theta_0 = \pi/8$ and $\phi_0 = 0$.

(ii) calculation, with the corresponding eigenvalues and eigenvectors, of the time dependent density matrix of Eqs. (7)–(9); (iii) calculation of the expectation value of the components of the total spin entering the definition of the squeezing parameter (10).

Fig. 1 displays the results obtained for the different factors $g(i, j)$ of Eq. (47) and for an initial coherent state with $\theta_0 = \pi/2$ and $\phi_0 = 0$. Although for practically all cases the results for ζ^2 are close to the canonical limit ($\zeta^2 = 1/2$), one sees fast oscillation of the actual values. It means that the time averaged value is much larger than the canonical limit. This behavior is easily understood because for $\theta_0 = \pi/2$, the averaged spin is perpendicular to the z direction. To investigate the sensitivity of the time dependence of ζ^2 upon the initial condition we have repeated the calculations for $\theta_0 = \pi/4$ and $\theta_0 = \pi/8$. The results are shown in Figs. 2 and 3, respectively. It is seen that, for smaller values of θ_0, the oscillations of ζ^2 are confined to a relatively narrow region, that is $0.7 < \zeta^2 < 1$ (see Fig. 3). For the limiting case $\theta_0 = 0$, one gets $\zeta^2 = 1$, as expected. From these results we can conclude that the time dependence of ζ^2 may be controlled by the spin–spin interactions as well as by the choice of the initial coherent state. At first glance, a proper choice of the coherent state may, indeed, minimize the time averaged value of ζ^2. The optimal values of ζ^2 are of the order of 0.7. From the results shown in Figs. 1–3 we can

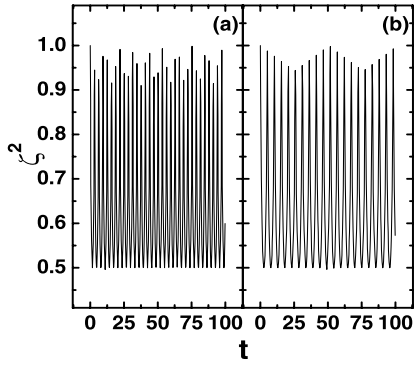

Fig. 4. Squeezing parameter, ζ^2, of Eq. (16) (inset (a)) and Eq. (19) (inset (b)). The curves have been obtained with the eigenvalues and eigenvector given in Eqs. (13) and (18), respectively for $N = 2$. The initial state is a coherent state (12) with $\theta_0 = \pi/2$ and $\phi_0 = 0$.

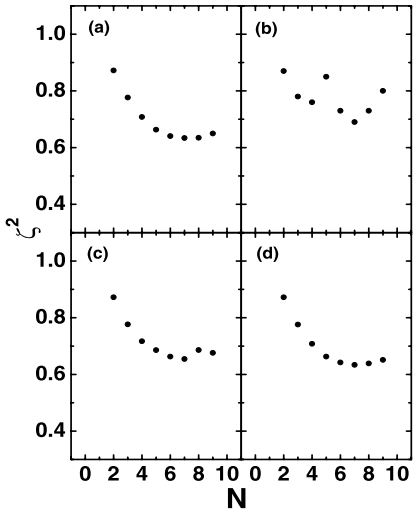

Fig. 5. Minimum value of the squeezing parameter, ζ^2, as a function of the number of atoms N, for different spin–spin interactions. Insets (a)–(d) show the results corresponding to the interactions of Eq. (47). We have chosen $\lambda_\perp = 1, \lambda_z = 2$, and the initial state is the coherent state of Eq. (12) with $\theta_0 = \pi/8$ and $\phi_0 = 0$. As explained in the text, to find the minima, we have studied the time evolution of ζ^2 in the interval $0 < t < 200$.

conclude that a spin chain with spin–spin interactions modulated by factors $g(i, j)$, like the ones of Eq. (47), may be regarded as an efficient device to keep information about the spin orientation and the relative strength of the total spin components.

For the sake of comparison, we show in Fig. 4 the results corresponding to Eqs. (16) and (19). While the results obtained with $g(i, j) = \sin|\frac{\pi}{2}(i - j)|$ (case (a)), show oscillations around the minima at $\zeta^2 = 1/2$, the ones obtained with $g(i, j) = e^{-\frac{1}{2}(i-j)^2}$ (case (b)) show, in addition, a modulation larger than in the previous case. A comparison with the curves displayed in Figs. 1(b) and 1(c) indicates that by increasing the number of atoms the oscillations are faster and minimum values of ζ^2 increase significantly.

We shall comment on the dependence of the squeezing parameter with N. Figs. 5–6 show the results for the squeezing parameter, ζ^2, as a function of the number of spin sites. The results shown in Fig. 5 are local minima, that is each of the points is the smallest value of ζ^2 calculated at a fixed value of N and in the same time interval of Figs. 1–3. The results shown in Fig. 6 are average values, over the time intervals, for the same interactions. To calculate the averages we have increased the time interval of Figs. 1–3 to maximum value $t = 1000$. As a general feature we observed that, relatively, smallest local and average minima in ζ^2 are obtained with

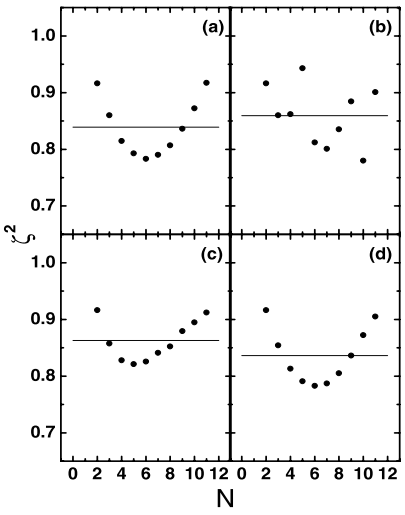

Fig. 6. Average (points) and their mean value (lines), of the squeezing factor ζ^2 as a function of the number of atoms, and for each of the interactions discussed in the text. The parameters used in the calculations are given in the captions to Fig. 5.

the interaction $g(i,j) = (\sin(\frac{\pi}{N}(i-j)))^{-2}$ [10–12]. This interaction, which represents a $1/r^2$-exchange, being r the distance between sites, has been proposed by Shastry [10] and by Haldane [11], in the study of a spin-1/2 anti-ferromagnetic Heisenberg chain. Lately, the spectrum of an isotropic s-1/2 Heisenberg chain was exactly solved using the same long-range interaction [12]. Considering the physical significance of Shastry and Haldane-interaction, the present results seemingly (but not conclusively) show that long-range interactions may optimize spin-squeezing devices.

4. Conclusions

To summarize, in this Letter we are reporting on the results of calculations of the spin-squeezing factor, ζ^2, in open s-1/2 pseudo-spin-chains with periodic and long-range interactions. We have investigated the dependence of ζ^2 on: (i) the interactions, (ii) the number of atoms, and (iii) the initial condition. The results, in general, are dependent also on the initial conditions, which in this work have been represented by spin coherent states. Although the number of atoms included in the calculations is relatively small ($N \leqslant 11$) the size of the configuration space was large enough to grasp the main tendency of the results. For the situation with a large number of atoms, we have discussed the use of the TDA approximation, which is exact in the subspace of spin up–spin down pairs. For this subspace, the results suggest the saturation of ζ^2. With all these limitations in mind, we may conclude by saying that pseudo-spin chains with long-range interactions may be optimal spin devices concerning the persistency of the squeezing factor ζ^2.

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