Solvable model for many-quark systems in QCD Hamiltonians

Tochtli Yépez-Martínez,[1] P. O. Hess,[1] A. P. Szczepaniak,[2,*] and O. Civitarese[3]
[1]*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Ciudad Universitaria, Circuito Exterior S/N, A.P. 70-543, 04510 México D.F. Mexico*
[2]*Department of Physics and Nuclear Theory Center, Indiana University Bloomington, Indiana, 47405-4202, USA*
[3]*Departamento de Física, Universidad Nacional La Plata C.C.67 (1900), La Plata, Argentina*
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Motivated by a canonical QCD Hamiltonian, we propose an effective Hamiltonian to represent an arbitrary number of quarks in hadronic bags. The structure of the effective Hamiltonian is discussed and the BCS-type solutions that may represent constituent quarks are presented. The single-particle orbitals are chosen as three-dimensional harmonic oscillators, and we discuss a class of exact solutions that can be obtained when a subset of single-particle basis states is restricted to include a certain number of orbital excitations. The general problem, which includes all possible orbital states, can also be solved by combining analytical and numerical methods.

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I. INTRODUCTION

One of the main interests in hadronic physics is to construct effective, low-energy approximations to QCD and to find methods enabling a theoretical treatment in its nonperturbative domain. Over the years, a number of nonrelativistic or semi-relativistic models describing quarks in hadronic bound states have been proposed. However, the full effect of quark-antiquark pairs has never been considered because such effects lead to a many-body problem that can only be treated in some approximation. In hadronic models, such approximations are typically driven by phenomenological considerations rather then QCD itself. Examples include, but are not limited to, the bag model [1,2], the constituent-quark model of Ref. [3], and algebraic models [4]. Within these phenomenological models a number of experimental data points have been considered (i.e., masses, form factors, structure functions, etc.), albeit models typically require a large number of parameters to reproduced them. Nevertheless, it is of interest to illuminate any possible connection between phenomenological models and QCD.

In this article, we explore the possible emergence of the constituent-quark picture from the many-body interactions inherent to QCD. In particular, we construct a simple effective quark Hamiltonian guided by QCD and examine various classes of solutions that may represent hadronic states. Because of the highly nonperturbative structure of QCD, we do not pretend that a simple Hamiltonian will be able to replace QCD. Nevertheless, it will exhibit the many-body aspects of QCD that are absent in other constituent models, with the advantage of leading to semi-analytic solutions (meaning that it might require simple numerical solutions of algebraic equations or diagonalization of a Hamiltonian matrix).

A possible connection between QCD and an effective quark-model Hamiltonian was presented in Ref. [5]. A confinement scenario and dynamical chiral symmetry breaking were discussed, but even in the chiraly broken phase the effective Hamiltonian remained complicated (i.e., it contains interactions among an infinite number of particles). Thus, in Ref. [6] a much-simplified version of the Hamiltonian was proposed and a solution for the SU(2)-color case was presented. In particular, quarks were assumed to be confined in single s-wave orbitals in a finite volume and no dynamical contributions from gluons was considered. The resulting effective Hamiltonian allows for analytic solutions with low energies saturated by color-neutral physical states and colored states shifted to arbitrarily high energies.

That study was followed by Ref. [7] where the limitation on the number of quark orbitals was relaxed and more analytic solutions were found.

In the present work, we extend the method of Refs. [6,7] to include all orbital levels in the quark sector. The color part is also extended to SU(3), whereas in the flavor sector, we keep the approximately mass-degenerate u and d quarks.

As in Ref. [6], dynamical contributions of gluons are not considered, and the system is confined to a finite volume. Dynamical gluon degrees of freedom can be added by expanding the model space. They will not be considered here, just like their absence in constituent models does not seem to spoil the successful phenomenology of the low mass states generated by the valence-quark dynamics alone. Under these approximations, we show how the QCD-motivated Hamiltonian used in this publication can be solved nearly analytically for a particular choice of the effective interaction between color charges. The main aim of this work is to show how to construct an algebraic model which potentially can describe the main features of the hadron spectrum. This will be shown in Sec. IV. The attractive feature of the model comes from its algebraic nature that, coupled with standard many-body techniques [8], makes it solvable. The rest of the article is organized as follows: In Sec. II, the model space and the Hamiltonian are defined. In Sec. III, we study the analytic solutions for two- and three-orbital-level systems in the chiral limit. This is followed by the consideration of an arbitrary number of levels. We study the quark-mass dependence and discuss why, in general, it is impossible to find simple analytic solutions unless an orthogonal transformation and a BCS-type of mean-field scheme is adopted. In Sec. IV we discuss the

*aszczepa@indiana.edu

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possible emergence of meson and baryon spectra that result from the lowest s- and p-quark orbitals. Conclusions are drawn in Sec. V.

II. THE HAMILTONIAN AND ITS MODEL SPACE

The construction of an effective QCD Hamiltonian that approximates its low-energy spectrum is based on the following assumptions: Because of confinement, the domain of fields is expected to be restricted to a finite volume in space, where individual hadrons are located. This is achieved by using the three-dimensional harmonic oscillator to describe single-particle levels. The confinement scale is then related to the width of the harmonic oscillator wave functions γ. Deconfinement transitions can then be studied as a limit when $\gamma \to 0$ since, in this basis, for a quark to probe large distances would require mixing with a large number of excited states. Alternatively, one can use a three-dimensional box but, as we shall show later, the oscillator representation is more convenient. Gluons are not taken into account as dynamical degrees of freedom; rather their affect is assumed to be accounted for by a static potential. Finally, to take advantage of algebraic techniques, the potential is smeared over the hadron scale and effectively replaced by a constant.

Most of the restrictions imposed in the model space in Refs. [6,7] are now removed. Therein we restricted quarks to be in SU(2) color but, as we shall see here, this restriction can be easily eliminated.

In Fig. 1, the model space for the quarks is schematically presented. Only the orbital states at positive energy are depicted, while vertical dashed lines separate columns of equal total spin j. As it will turn out, the Hamiltonian can be written in terms of operators that act only within each column. This will greatly simplify the task of finding an analytic procedure.

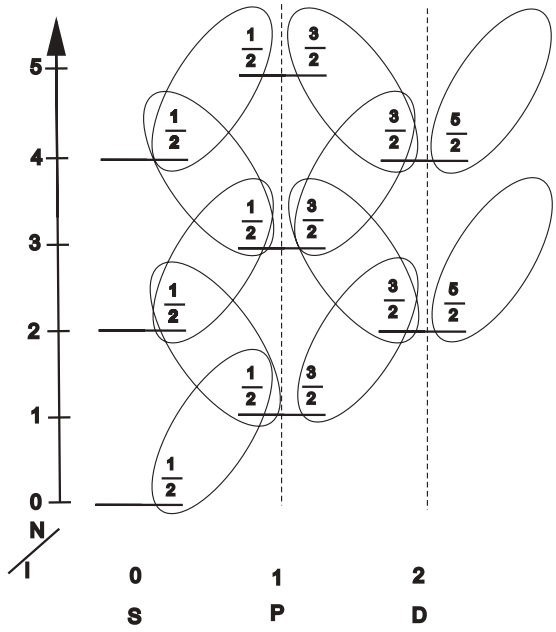

FIG. 1. Illustration of the effect of the Hamiltonian on the basis states. The operators, which appear in the kinetic energy, can be divided into columns of operators related to a given spin j, and which commute with those of different columns. Only the positive energy states are plotted. The ellipses indicate which states are connected through the interaction terms appearing in the kinetic energy. Columns of different j are separated by a vertical dashed line.

The fermion creation operator for a quark in the level α ($\alpha = \pm\frac{1}{2}$ denotes the upper or lower Dirac level) with angular momentum l coupled with quark spin-$\frac{1}{2}$ to total spin j, is given by

$$\boldsymbol{b}^{\dagger}_{\alpha(N,l\frac{1}{2})j\lambda,cf} = \sum_{m\sigma} \left(lm, \frac{1}{2}\sigma \middle| j\lambda \right) \boldsymbol{b}^{\dagger}_{\alpha Nlm,\sigma cf}. \tag{1}$$

The remaining indices N, λ, c, and f refer to the oscillator number, spin, color, and flavor indices, respectively. The factor $(lm, \frac{1}{2}\sigma|j\lambda)$ is an SU(2) Clebsch-Gordan coefficient. The quarks are in the fundamental representation of SU(2) in spin- and flavor-spin and SU(3) for the color part representation.The creation operator on the right-hand side corresponds to a spin-orbit decoupled basis. The operator with upper-level indices is defined in the standard way (see Eq. (1)):

$$\begin{aligned}\boldsymbol{b}^{\dagger\alpha(N,l,\frac{1}{2})j\lambda,cf} \equiv{}& (-1)^{\frac{1}{2}-\alpha}(-1)^{j-\lambda}(-1)^{\frac{Y_C}{2}+T_{Cz}}(-1)^{\frac{1}{2}-f} \\ &\times \boldsymbol{b}^{\dagger}_{-\alpha(N,l,\frac{1}{2})j-\lambda,\bar{c}-f}.\end{aligned} \tag{2}$$

Here and in the following, the capital index C denotes globally the color part. The phase convention is taken from Ref. [9] (with a corrected sign in front of T_z). The lower-case c denotes the magnetic quantum numbers of the color part and is a short-hand notation for (Y_C, T_C, T_{Cz}), with Y_C being the color hypercharge, T_C the color isospin, and T_{Cz} its third component. The bar over the index c refers to change from $c = (Y_C, T_C, T_{Cz})$ to $\bar{c} = (-Y_C, T_C, -T_{Cz})$ (the conjugate index). In the following, we abbreviate $(-1)^{\frac{Y_C}{2}+T_C}$ as $(-1)^{\chi_c}$.

When the index α has the value $\alpha = +\frac{1}{2}$ (upper level), the operators $\boldsymbol{b}$ are replaced by $\boldsymbol{a}$, and are referred to as quark operators. In contrast, when $\alpha = -\frac{1}{2}$ we define the operators as $\boldsymbol{d}$ and as antiquark operators. The $\boldsymbol{b}$ creation and annihilation operators are then converted into the $\boldsymbol{d}$ annihilation and creation operators. This notation exploits the Dirac formulation for fermions, where a fermion in an upper level is denoted as a particle while a fermion hole in the lower level is denoted as an antiparticle. These and their conjugates will be used to express the effective Hamiltonian. The kinetic Dirac energy is originally given by

$$\mathbf{K} = \int d\boldsymbol{x}\, \boldsymbol{\psi}^{\dagger}(\boldsymbol{x})\,(-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + \beta m_0)\,\boldsymbol{\psi}(\boldsymbol{x}). \tag{3}$$

A mass term has been included, in contrast with Ref. [7], and the potential term is given by

$$\boldsymbol{V} = \int d\boldsymbol{x}\, d\boldsymbol{y}\, \boldsymbol{\psi}^{\dagger}(\boldsymbol{x}) T^a \boldsymbol{\psi}(\boldsymbol{x}) V(|\boldsymbol{x}-\boldsymbol{y}|) \boldsymbol{\psi}^{\dagger}(\boldsymbol{y}) T^a \boldsymbol{\psi}(\boldsymbol{y}). \tag{4}$$

The static potential $V(|\boldsymbol{x}-\boldsymbol{y}|)$ simulates the gluon-quark interaction. The origin of this expression is the Faddeev-Popov term of the QCD Hamiltonian [10]. It is essentially a color-color Casimir interaction and when averaged over the confined hadronic bag it is replaced by a constant V_0 and becomes exactly the total-color operator [6]. This can easily

be seen by noting that $\int d\boldsymbol{x}\,\boldsymbol{\psi}^{\dagger}(\boldsymbol{x})T^{a}\boldsymbol{\psi}(\boldsymbol{x})$ represents the color charge and a constant potential separates the integrals over $\boldsymbol{x}$ and $\boldsymbol{y}$.

In what follows, we will first consider the potential interaction, because it will result in a very simple spectrum that can later be used to diagonalize the hopping interactions brought by the kinetic energy.

To do so, we expand the fermion fields $\boldsymbol{\psi}^{\dagger}$ and $\boldsymbol{\psi}$ in a complete particle basis in terms of the decoupled fermion creation and annihilation operators:

$$\boldsymbol{\psi}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{\psi}_1(\boldsymbol{x},\sigma,c,f) \\ \boldsymbol{\psi}_2(\boldsymbol{x},\sigma,c,f) \end{pmatrix},$$

with

$$\begin{aligned} \boldsymbol{\psi}_1(\boldsymbol{x},\sigma,c,f) &= \sum_{Nlm} \boldsymbol{b}_N^{\frac{1}{2}lm,\sigma,c,f} R_{Nl}(|\vec{x}|)Y_{lm}(\Omega_x)\chi_\sigma, \\ \boldsymbol{\psi}_2(\boldsymbol{x},\sigma,c,f) &= \sum_{Nlm} \boldsymbol{b}_N^{-\frac{1}{2}lm,\sigma,c,f} R_{NL}(|\vec{x}|)Y_{lm}(\Omega_x)\chi_\sigma. \end{aligned} \tag{5}$$

Here, Ω_x denotes the angular components θ_x and ϕ_x of the position vector $\mathbf{x}$. The fields have components in spin, flavor, and color. In Eqs. (3) and (4) for the kinetic and potential energy, the sum over the spin, color, and flavor indices is implicit.

A. The potential

By substituting Eq. (5) into Eq. (4), and transforming to the coupled-fermion creation and annihilation operators as defined in Eq. (1), we are led to the following form for the potential interaction (see Appendix A for details):

$$\begin{aligned} \boldsymbol{V} = \sum_X & \sqrt{(2j_1+1)(2j_3+1)}V(N_i,l_i,j_i,L) \\ & \times (-1)^M(-1)^{\frac{Y_C}{2}+T_{C_z}}\big[\boldsymbol{b}^{\dagger}_{\alpha(N_1,l_1,\frac{1}{2})j_1\lambda_1,c_1f}\boldsymbol{b}^{\alpha,f}_{(N_2,l_2,\frac{1}{2})j_2\lambda_2,c_2} \\ & \times \langle j_1\lambda_1, j_2\lambda_2|LM\rangle\langle(1,0)c_1,(0,1)c_2|(1,1)C\rangle_1\big] \\ & \times \big[\boldsymbol{b}^{\dagger}_{\alpha',(N_3,l_3,\frac{1}{2})j_3,\lambda_3,c_3f'}\boldsymbol{b}^{\alpha',f'}_{(N_4,l_4,\frac{1}{2})j_4\lambda_4,c_4} \\ & \times \langle j_3\lambda_3, j_4\lambda_4|L-M\rangle\langle(1,0)c_3,(0,1)c_4|(1,1)\bar{C}\rangle_1\big], \end{aligned} \tag{6}$$

where X stands for the collection of quantum numbers N_i, j_i, λ_i, l_i, c_i, α, f, α', f', L, M, and C. The SU(3) Clebsch-Gordan coefficients carry a multiplicity subindex 1 because the fermion creation and annihilation operators are coupled to the generators of the color SU(3). The coupling corresponds to the $(1,0)\otimes(0,1) = (0,0)+(1,1)$ decomposition of the product of two quark representations in which no multiplicity appears. The convention is taken from Refs. [9,11,12] [with a corrected sign in the SU(3) phase]. As before, the indices α and α' are used to label pseudo spin components and l_i, f, c_i, j_i, λ_i label orbital angular momentum, flavor, and color components, total spin, and its projection, respectively. The quantities $V(N_i,l_i,j_i,L)$ define the intensity of each component of the interaction in Eq. (4) and are given (before space averaging) by

$$\begin{aligned} & V(N_i,l_i,j_i,L) \\ & = \frac{(-1)^{j_2+\frac{1}{2}+j_4+\frac{1}{2}}}{(2L+1)}\left(\frac{3}{2}\right)\int |\boldsymbol{x}|^2 d|\boldsymbol{x}||\boldsymbol{y}|^2 d|\boldsymbol{y}| R_{N_1l_1}(|\boldsymbol{x}|) \\ & \quad \times R_{N_2l_2}(|\boldsymbol{x}|)R_{N_3l_3}(|\boldsymbol{y}|)R_{N_4l_4}(|\boldsymbol{y}|) \\ & \quad \times \int_{-1}^{1} d(\cos\theta)P_L(\cos\theta)V(|\boldsymbol{x}|,|\boldsymbol{y}|,\cos\theta) \\ & \quad \times \frac{\prod_{i=1}^{4}\sqrt{(2l_i+1)(2j_i+1)}\langle l_1 0, l_2 0|L0\rangle\langle l_3 0, l_4 0|L0\rangle}{\sqrt{(2j_1+1)(2j_3+1)}} \\ & \quad \times \begin{Bmatrix} j_1 & l_1 & \frac{1}{2} \\ l_2 & j_2 & L \end{Bmatrix}\begin{Bmatrix} j_3 & l_3 & \frac{1}{2} \\ l_4 & j_4 & L \end{Bmatrix}. \end{aligned} \tag{7}$$

The factor $\frac{3}{2}$ originates in the triple-reduced matrix elements of the color operator, as explained in Appendix A, and θ is the polar angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

The expansion in partial waves simplifies considerably how the two-body interaction kernel $V(|\mathbf{x}-\mathbf{y}|)$ is approximated by its spacial average over the hadronic bag, $V(|\mathbf{x}-\mathbf{y}|)\to V_0$. Then, the only nonvanishing contribution comes from $L=0$ which, as a consequence, separates the integral over $\boldsymbol{x}$ and $\boldsymbol{y}$ and leads to $V(N_i,l_i,j_i,L)$ being

$$V(N_i,l_i,j_i,L=0) = \frac{V_0}{2}\delta_{L,0}\delta_{N_1N_2}\delta_{j_1j_2}\delta_{l_1l_2}\delta_{N_3N_4}\delta_{j_3j_4}\delta_{l_3l_4}. \tag{8}$$

Thus, for the potential term in the Hamiltonian we finally have

$$V = \frac{V_0}{2}\mathcal{C}_2[SU(3)], \tag{9}$$

where $\mathcal{C}_2[SU(3)]$ is the second order Casimir operator of SU(3) color and is given by

$$\begin{aligned} \mathcal{C}_2[SU(3)] &= \frac{3}{2}\sum_{c_1c_2}\boldsymbol{C}^{c_2}_{c_1}\boldsymbol{C}^{c_1}_{c_2}, \\ \boldsymbol{C}^{c_2}_{c_1} &= (\boldsymbol{b}^{\dagger}_{c_1}\cdot\boldsymbol{b}^{c_2}) - \frac{\delta_{c_1c_2}}{3}\boldsymbol{N}, \\ (\boldsymbol{b}^{\dagger}_{c_1}\cdot\boldsymbol{b}^{c_2}) &= \sum_{\alpha Nlj\lambda f}\boldsymbol{b}^{\dagger}_{\alpha(Nl,\frac{1}{2})j\lambda c_1 f}\boldsymbol{b}^{\alpha(Nl,\frac{1}{2})j\lambda c_2 f}, \\ \boldsymbol{N} &= \sum_{\alpha Nlj\lambda cf}\boldsymbol{b}^{\dagger}_{\alpha(Nl,\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\alpha(Nl,\frac{1}{2})j\lambda cf}. \end{aligned} \tag{10}$$

The $\boldsymbol{C}^{c_2}_{c_1}$ are the generators of the SU(3)-color group. Its eigenvalue is given by [9]

$$\lambda_C^2 + \lambda_C\mu_C + \mu_C^2 + 3\lambda_C + 3\mu_C, \tag{11}$$

with (λ_C,μ_C) defining the SU(3) irreducible representation (irrep). Thus, in this approximation the effect of color-charge interactions is to separate colored from non-colored states without any regard to spacial distribution of color. The second-order Casimir operator is equivalent to the color-spin in SU(2) color discussed in Ref. [6].

B. The kinetic energy

The spacial dependence on the quark distribution is brought by the kinetic energy term. We first consider the mass term in Eq. (3). It leads to an operator proportional to the sum of the quark and antiquark number operators:

$$\begin{aligned} & m_0(\boldsymbol{n}_q + \boldsymbol{n}_{\bar{q}}) + m_0 \sum_{Nljf\lambda cf} 1 \\ & = m_0 \sum_{Nj\lambda cf} \Big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & \quad - \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \Big] \\ & \quad + m_0 \sum_{N'j\lambda cf} \Big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & \quad - \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \Big]. \end{aligned} \tag{12}$$

The terms with $\alpha = \frac{1}{2}$ count the number of quarks in the upper level, whereas the terms with $\alpha = -\frac{1}{2}$ together with the last term count the number of holes, thus the number of antiquarks, in the lower level. The last term on the left-hand side is a constant and may be skipped if convenient. The expressions $\boldsymbol{n}_q = \boldsymbol{b}^{\dagger}_{\frac{1}{2}\nu} \boldsymbol{b}^{\frac{1}{2}\nu} = \boldsymbol{a}^{\dagger}_{\nu} \boldsymbol{a}^{\nu}$ and $\boldsymbol{n}_{\bar{q}} = \boldsymbol{b}^{-\frac{1}{2}\nu} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}\nu} = \boldsymbol{d}^{\dagger\nu} \boldsymbol{d}_{\nu}$ are the quark and antiquark number operators, respectively. The index ν is a short-hand notation for all the indices that label individual creation and annihilation operators.

The momentum-dependent part of the kinetic energy, when expressed it in terms of the fermion creation and annihilation operators, is given by (see Appendix for derivation)

$$\boldsymbol{K} = (\widetilde{\boldsymbol{K}}_+ + \widetilde{\boldsymbol{K}}_-), \tag{13}$$

where

$$\begin{aligned} \widetilde{\boldsymbol{K}}_+ \equiv \sqrt{\gamma} \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{\lambda cf} \Bigg[& \left(\frac{N-j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \\ & \times \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \left(\frac{N+j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \left(\frac{N-j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \left(\frac{N+j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \Bigg], \\ \widetilde{\boldsymbol{K}}_- \equiv \sqrt{\gamma} \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{\lambda cf} \Bigg[& \left(\frac{N-j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \\ & \times \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \left(\frac{N+j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \left(\frac{N-j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \left(\frac{N+j+\frac{3}{2}}{2} \right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \Bigg]. \end{aligned} \tag{14}$$

The $\widetilde{\boldsymbol{K}}_+$ operator shifts quarks from the lower ($\alpha = -\frac{1}{2}$) to the upper ($\alpha = +\frac{1}{2}$) level, whereas $\widetilde{\boldsymbol{K}}_-$ does the opposite. The tilde is used to distinguish the above operators from actual generators of an SU(2) algebra.

The mass term will be skipped in most of the cases discussed here for the reason that the kinetic energy part without the mass term can be identified as being proportional to a component of a generator on an SU(2) algebra. Thus, it can be diagonalized exactly. The mass term, however, does not commute with the $\widetilde{\boldsymbol{K}}_m$ operators, which determines the momentum dependence of kinetic energy and destroys the possibility of exact diagonalization. That they do not commute can also be seen by noting that, for example, the $\widetilde{\boldsymbol{K}}_+$ operator creates a particle-hole pair that is equivalent, as just noted, to raising the number of quarks and antiquarks. Nevertheless, when we treat the complete problem, although the SU(2) structure is lost, we can still diagonalize the kinetic energy part by using the BCS formalism [8].

III. ANALYTIC AND SEMI-ANALYTIC SOLUTIONS

In what follows, we shall consider two, three, and finally an arbitrary number of single-quark orbital levels. The reason for choosing this path is to investigate the analytical properties of the solutions and to see if any features in the small-basis approximation can be generalized to the case when any number of quark levels is allowed. In the two- and three-level case, the mass term is neglected, otherwise no simple solution can be obtained. The mass term will be included when all orbital levels are taken into account because, as we show below, in this limit the mass term can be treated together with kinetic energy within the BCS formalism.

A. The two level-systems

This refers to the case when, for a given j, two orbital levels are considered, one with N oscillator quanta and with orbital angular momentum $l = j + \frac{1}{2}$ and the other with either $N' = (N-1)$ or $N' = (N+1)$ quanta and orbital angular momentum $l = j - \frac{1}{2}$. For example, for $N = 1$ and $j = \frac{1}{2}$, this corresponds to the lowest p- plus the lowest or next-to-lowest s-orbital.

The momentum-dependent part of the kinetic energy can be related to $\boldsymbol{K}_{\pm}$ operators. These operators are proportional to $\widetilde{\boldsymbol{K}}_m$, when restricted to two orbital levels, and are of the form

$$\begin{aligned} \boldsymbol{K}^{jN}_{+} = \sqrt{2 \cdot 8} \sqrt{2j+1} \Big\{ & \Big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,l+1,\frac{1}{2})j} \otimes \boldsymbol{b}_{\frac{1}{2}(N',l,\frac{1}{2})j} \Big]^{000}_{000} \\ & - \Big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',l,\frac{1}{2})j} \otimes \boldsymbol{b}_{\frac{1}{2}(N,l+1,\frac{1}{2})j} \Big]^{000}_{000} \Big\}, \end{aligned}$$

$$\boldsymbol{K}_{-}^{jN} = \sqrt{2\cdot 8}\sqrt{2j+1}\Big\{-\big[\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,l+1,\frac{1}{2})j}\otimes \boldsymbol{b}_{-\frac{1}{2}(N',l,\frac{1}{2})j}\big]^{000}_{000} + \big[\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',l,\frac{1}{2})j}\otimes \boldsymbol{b}_{-\frac{1}{2}(N,l+1,\frac{1}{2})j}\big]^{000}_{000}\Big\}. \quad (15)$$

The spin-color-flavor coupling denoted by "⊗" is defined as

$$[\boldsymbol{a}^{\Gamma_1}\otimes \boldsymbol{b}^{\Gamma_2}]^{\Gamma}_{\mu} = \sum_{\mu_1\mu_2}\langle \Gamma_1\mu_1,\Gamma_2\mu_2|\Gamma\mu\rangle A^{\Gamma_1}_{\mu_1}B^{\Gamma_2}_{\mu_2}. \quad (16)$$

Here, Γ_k and Γ stand for the quantum numbers denoting combined representation of the spin, color, and flavor, and μ_k and μ are a compact notation for the magnetic quantum numbers (e.g., $\mu=\lambda cf$). The expression of Eq. (15) can be easily derived from Eq. (14) by noting that the coupling to spin-color-flavor singlet implies a contraction of all indices. The factor $\sqrt{2\cdot 8\cdot(2j+1)}$ is the square root of the multiplicities of the flavor, color, and spin representation, respectively, and yields the correct normalization of the singlet representation.

The first index in the creation and annihilation operators refers to the pseudo-spin component $\frac{1}{2}$ for the upper level, and $-\frac{1}{2}$ for the lower level. Thus, when $l=1$ the kinetic energy couples quarks in the s level with quarks in the p level of total spin $j=\frac{1}{2}$. When $l=2$, quarks in the p level with total spin $j=\frac{3}{2}$ are coupled by the kinetic term to quarks in the d level, also with total spin $\frac{3}{2}$. The orbital values l are given by $j\pm\frac{1}{2}$. The radial number N starts from $j+\frac{1}{2}$ and acquires odd (even) values, while N' has even (odd) values only and satisfies $N'=N-1$ or $N+1$. This selection rule is strictly obeyed only for the harmonic oscillator. Note that the $\boldsymbol{K}_m$ operators are coupled, for each given spin j, to total color zero. Thus, they commute with the total color-spin operator of each column j.

The $\boldsymbol{K}$ operators satisfy the commutation relations

$$[\boldsymbol{K}_{+}^{jN},\boldsymbol{K}_{-}^{jN}] = 2\boldsymbol{K}_{0}^{jN}, \quad [\boldsymbol{K}_{0}^{jN},\boldsymbol{K}_{\pm}^{jN}] = \pm\boldsymbol{K}_{\pm}^{jN}, \quad (17)$$

so that for each combination of N and j they form an $\mathrm{SU}^{jN}(2)$ pseudo-spin group. Furthermore, operators with different spins commute. The operator $\boldsymbol{K}_0^{jN}$ is given by

$$\boldsymbol{K}_{0}^{jN} = \tfrac{1}{2}\big[\big(\boldsymbol{N}_{\frac{1}{2}}^{jN}+\boldsymbol{N}_{\frac{1}{2}}^{jN'}\big)-\big(\boldsymbol{N}_{-\frac{1}{2}}^{jN}+\boldsymbol{N}_{-\frac{1}{2}}^{jN'}\big)\big], \quad (18)$$

which is to say, by half of the difference between the number of quarks in the upper and lower level. The quarks are coupled to total spin j. Note that the pseudo-spin group $\mathrm{SU}^{jN}(2)$ defined by the $\boldsymbol{K}_m$ operators is different from the pseudo-spin group in the original definition, which has been given with reference to lowering and raising operators within the same orbital. The $\boldsymbol{K}_m$ operators, defined here, raise and lower quarks from one orbital to a different orbital.

The relation to the operators in Eq. (14) is obtained by either (a) restricting to the combination of the orbital N, $l=j+\frac{1}{2}$ with the orbital $(N-1)$, $l=j-\frac{1}{2}$, or (b) to the combination of the orbital N, $l=j+\frac{1}{2}$ and the orbital $(N+1)$, $l=j-\frac{1}{2}$. The relation of the kinetic-energy term and the generators of the SU(2) algebra is then given by

$$\widetilde{\boldsymbol{K}}_{\pm}^{jN} = \sqrt{\frac{\gamma\big(N-j+\frac{3}{2}\big)}{2}}\boldsymbol{K}_{\pm}^{jN} \equiv A_{aNj}\boldsymbol{K}_{\pm}^{jN} \quad (19)$$

for the case (a), and

$$\widetilde{\boldsymbol{K}}_{\pm}^{jN} = \sqrt{\frac{\gamma\big(N+j+\frac{3}{2}\big)}{2}}\boldsymbol{K}_{\pm}^{jN} \equiv A_{bNj}\boldsymbol{K}_{\pm}^{jN} \quad (20)$$

for the case (b).

The operators $\boldsymbol{K}_m$ commute with the total-color operator in each column j and, thus, also with the total-color operator. This is because for each given j they are coupled to color zero. As a consequence, the Hamiltonian can be diagonalized analytically. It is given by

$$\boldsymbol{H}^{jN} = A_{\kappa Nj}(\boldsymbol{K}_{+}^{jN}+\boldsymbol{K}_{-}^{jN}) + \frac{V_0}{2}\boldsymbol{\mathcal{C}}_2[\mathrm{SU}(3)], \quad (21)$$

with $A_{\kappa Nj}$ ($\kappa=a,b$), defined in Eqs. (19) and (20). The expression in the parentheses of the first term is just $\boldsymbol{K}_x^{jN}$, whose eigenvalues are known. The kinetic part is then just $2\mathbf{K}_x$. Thus, the eigenvalues of this Hamiltonian are

$$E^{jN} = 2A_{\kappa Nj}M_J + \frac{V_0}{2}\big(\lambda_C^2+\lambda_C\mu_C+\mu_C^2+3\lambda_C+3\mu_C\big), \quad (22)$$

where M_J is the projection of the pseudo-spin operator within the pseudo-spin J onto the x axis and (λ_C,μ_C) define the irreducible representations (irreps) of the SU(3) color group [9,11]. When color is zero, the irrep is given by (0, 0), whereas for a color octet it is (1, 1), etc. We used the definition of the second order Casimir operator as given in Refs. [9,11]. For colorless states, this reduces to the simple formula $2A_{\kappa Nj}M_J$.

Figure 1 illustrates the effect of the kinetic term discussed above, without showing the negative-energy levels The negative-energy levels are just a copy of the levels at positive energy inverted to negative energy of what is shown in Fig. 1. Each of the ellipses represents a combination of orbitals for which the analytic solution given above in Eq. (22) applies. The orbitals, which the ellipses connect, are not only the ones at positive energy, as shown in the figure, but also represent the connection to the negative-energy levels.

B. The three-level system

The three-level system consists of the following levels: We add to each level, given by j, N, $l=j+\frac{1}{2}$, two levels with $l=j-\frac{1}{2}$ and $(N-1)$ and $(N+1)$ oscillator quanta. The kinetic energy for this three-level system becomes

$$\widetilde{\boldsymbol{K}} = \widetilde{\boldsymbol{K}}_{+}^{jN}+\widetilde{\boldsymbol{K}}_{-}^{jN}, \quad (23)$$

where

$$\widetilde{\boldsymbol{K}}_{+}^{jN} = \sqrt{\gamma}\sum_{\lambda cf}\Bigg[\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \times \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} + \left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf}$$

$$
\begin{aligned}
&+\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \\
&+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\Bigg], \\
\widetilde{\boldsymbol{K}}^{jN}_{-} &= \sqrt{\gamma}\sum_{\lambda cf}\Bigg[\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \\
&\times \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\
&+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\
&+\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \\
&+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\Bigg].
\end{aligned}
\tag{24}
$$

Note that this is exactly the same expression as in Eq. (14) but without summations over j and N.

Defining

$$
\boldsymbol{K}^{jN}_{\pm} = \eta\widetilde{\boldsymbol{K}}^{jN}_{m}, \quad \boldsymbol{K}^{jN}_{0} = \eta^{2}\widetilde{\boldsymbol{K}}^{jN}_{0}, \tag{25}
$$

it can be verified that, by choosing the factor η appropriately, the operators $\boldsymbol{K}^{jN}_{m}$, just as in the two-level case, show the standard form of an SU(2) algebra, such that $[\boldsymbol{K}^{jN}_{+}, \boldsymbol{K}^{jN}_{-}] = \boldsymbol{K}^{jN}_{0}$ and $[\boldsymbol{K}^{jN}_{0}, \boldsymbol{K}^{jN}_{\pm}] = \pm\boldsymbol{K}^{jN}_{\pm}$ (see Appendix D for details). In the three-level case, the result then resembles the one in Eq. (21), including the expression for the energy given by Eq. (22). This comes as a complete surprise because it is not obvious at all that an SU(2) structure is contained in the relativistic kinetic term in this quasiparticle basis.

The fact that we found, even in such a complicated system, an analytic solution is evidence that probably a complete analytical treatment of the whole problem (i.e., including all orbital levels), might be possible, excepting a simple numerical solution of a set of equations. We discuss this case in the following section.

C. An arbitrary number of levels

The kinetic energy that is not restricted to operate in a specific subspace of single-particle orbitals is given by Eq. (14). The two parts of this kinetic energy; the one that moves quarks to the upper level and the one that moves quarks to the lower level, no longer satisfy an SU(2) algebra. We will also add a mass term, which by itself destroys any SU(2) structure once present (see discussion in Sec. II B). Nevertheless, we can exploit the structure encountered in the two- and three-level case to simplify the problem.

By noting that the total kinetic energy is given by

$$
\boldsymbol{K} = \sum_{j}\boldsymbol{K}^{j}, \tag{26}
$$

we can select one particular value j and solve the problem for it. The complete solution is the sum of the solutions for all j. We introduce a cutoff value n for N, which can be chosen arbitrarily. Therefore,

$$
\begin{aligned}
\boldsymbol{K}^{(j,n)} = &\sum_{N=j+\frac{1}{2}}^{n,\Delta N=2\,\min(n,N+1)}\sum_{N'=N-1}\sum_{\lambda cf} k^{j}_{NN'} \\
&\times\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\right. \\
&\left.+\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\right] \\
&+\sum_{N=j+\frac{1}{2}}^{n,\Delta N=2\,\max(n,N+1)}\sum_{N'=N-1}\sum_{\lambda cf} k^{j*}_{NN'} \\
&\times\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})\lambda cf}\right. \\
&\left.+\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\right] \\
&+m_{0}\sum_{N=j+\frac{1}{2}}^{n}\sum_{\lambda cf}\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\right. \\
&\left.-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\right] \\
&+m_{0}\sum_{N'=j-\frac{1}{2}}^{n}\sum_{\lambda cf}\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\right. \\
&\left.-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\right].
\end{aligned}
\tag{27}
$$

In the last two terms, the step size of N and N' is 2. We have skipped a trivial constant at the end of Eq. (27), which will be included in the BCS formalism that we treat later. The factors $k^{j}_{NN'}$ can be read from Eq. (14).

The kinetic energy can now be rewritten as

$$
\begin{aligned}
\boldsymbol{K}^{(j,n)} = &\sum_{N=j+\frac{1}{2}}^{\Delta N=2,n\,\min(n,N+1)}\sum_{N'=N-1}\sum_{\lambda cf}\left|k^{j}_{NN'}\right| \\
&\times\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\right. \\
&\left.+\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\right] \\
&+\sum_{N=j+\frac{1}{2}}^{\Delta N=2,n\,\min(n,N+1)}\sum_{N'=N-1}\sum_{\lambda cf}\left|k^{j}_{NN'}\right| \\
&\times\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})\lambda cf}\right. \\
&\left.+\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\right] \\
&+m_{0}\sum_{N=j+\frac{1}{2}}^{\Delta N=2,n}\sum_{\lambda cf}\left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\right. \\
&\left.-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\right]
\end{aligned}
$$

$$+m_0\sum_{N'=j-\frac{1}{2}}^{\Delta N'=2,n}\sum_{\lambda cf}\Big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\Big], \tag{28}$$

whereas the mass term is unaffected. Next, we apply a unitary transformation

$$\begin{aligned}\boldsymbol{b}^{\dagger}_{\pm\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}&=\sum_k\alpha^*_{jNk}\widehat{\boldsymbol{b}}^{\dagger}_{\pm\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf},\\ \boldsymbol{b}^{\dagger}_{\pm\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}&=\sum_q\beta^*_{jN'q}\widehat{\boldsymbol{b}}^{\dagger}_{\pm\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}.\end{aligned} \tag{29}$$

The new operators have to also satisfy the fermion anticommutation rules, which impose a constriction on α and β. In particular, for the operators with orbital spin $l=j+\frac{1}{2}$, we have

$$\begin{aligned}\Big\{\boldsymbol{b}^{\dagger}_{\pm\frac{1}{2}(N_1,j+\frac{1}{2},\frac{1}{2})j\lambda cf},\boldsymbol{b}^{\pm\frac{1}{2}(N_2,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\Big\}&=\delta_{N_1N_2}\\ \Longrightarrow\sum_k\alpha^*_{jN_1k}\alpha_{jN_2k}&=\delta_{N_1N_2}.\end{aligned} \tag{30}$$

whereas for $l=j-\frac{1}{2}$,

$$\begin{aligned}\Big\{\boldsymbol{b}^{\dagger}_{\pm\frac{1}{2}(N'_1,j-\frac{1}{2},\frac{1}{2})j\lambda cf},\boldsymbol{b}^{\pm\frac{1}{2}(N'_2,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\Big\}&=\delta_{N'_1N'_2}\\ \Longrightarrow\sum_q\beta^*_{jN'_1q}\beta_{jN'_2q}&=\delta_{N_1N_2}.\end{aligned} \tag{31}$$

A few words are necessary concerning the variables α_{jNk} and $\beta_{jN'q}$. We cut the space at a given $N=N_{\max}=n$ for $j+\frac{1}{2}$ such that the maximal N' equals $N_{\max}-1$. Choosing a given integer n as the maximal number for N, the range of k and q is from 1 to n_j (see Fig. 2), where n_j is the number of levels for $l=j\pm\frac{1}{2}$. For example, in the two-level system and $j=\frac{1}{2}$, we consider one p orbital (orbital spin $j+\frac{1}{2}$ and $N=1$) and one s orbital (orbital spin $j-\frac{1}{2}$ and $N'=0$). In this case, $n_j=1$ and $k=q=1$. By using the symmetry of the matrices, we get for α_{jNk} a total of $n_j(n_j+1)/2$ linear independent elements, with the same result for β'_{jNq}. In order to determine all of them, we need $n_j(n_j+1)$ conditions, as we shall explain later. Applying these transformations, the kinetic energy reads

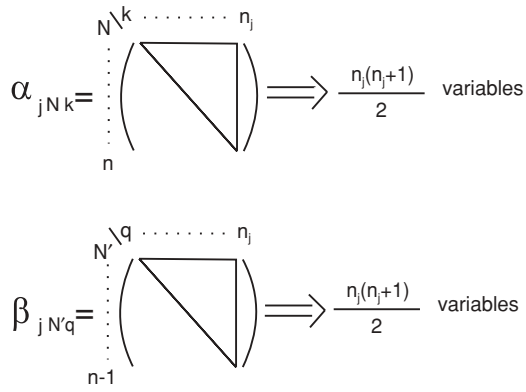

FIG. 2. Structure of the matrices α and β. The n_j give the dimension of the matrices. Although N goes from $j+\frac{1}{2}$ up to n (i.e., the maximal number of N), considering that N takes steps of two, the number of rows is exactly n_j. The relationship between n_j and n is given by $n_j=\frac{1}{2}[n-(j+\frac{1}{2})]+1$.

$$\begin{aligned}\boldsymbol{K}^j=&\sum_{kq}\sum_{\lambda cf}\widetilde{k}^j_{kq}\Big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\\ &+\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\Big]\\ &+\sum_{kq}\sum_{\lambda cf}\widetilde{k}^j_{kq}\Big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\\ &+\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\Big]\\ &+\sum_{k}\sum_{\lambda cf}m_{0,k,j+\frac{1}{2}}\Big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\\ &-\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\Big]\\ &+\sum_{q}\sum_{\lambda cf}m_{0,k,j-\frac{1}{2}}\Big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\\ &-\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\Big].\end{aligned} \tag{32}$$

with

$$\begin{aligned}m_{0,k,j-\frac{1}{2}}&=\sum_{N'=0,2,4,\ldots}^{n-1}m_0|\beta_{jN'k}|^2,\\ m_{0,k,j+\frac{1}{2}}&=\sum_{N=1,3,5,\ldots}^{n}m_0|\alpha_{jNk}|^2,\end{aligned} \tag{33}$$

and the factors $\widetilde{k}^j_{kq}$ being given by

$$\widetilde{k}^j_{kq}=\sum_{N=j+\frac{1}{2}}^{\Delta N=2,n}\sum_{N'=N-1}^{\min(n,N+1)}\big|k^j_{NN'}\big|\alpha^*_{jNk}\beta_{jN'q}. \tag{34}$$

In order that, for a given j value, the kinetic energy be diagonal in the orbital index, we require

$$\widetilde{k}^j_{kq}=0\quad\text{for}\quad k\neq q. \tag{35}$$

These leads to $n_j(n_j-1)$ conditions that, together with the $2n_j$ normalization conditions $\sum_k|\alpha_{jNk}|^2=1$ and $\sum_q|\beta_{jN'q}|^2=1$, leads to the $n_j(n_j+1)$ equations required to determine all variables α_{jNk} and $\beta_{jN'q}$.

It shows that, at the end, when all restrictions are fulfilled, we arrive at new orbitals labeled by the index $k=1,2,\ldots,n_j$, which are divided into particles and antiparticles. Note that, when only the diagonal components of $\widetilde{k}^j_{kq}$ are different from zero, the kinetic energy contributes to n_j new orbitals for each orbital angular momentum $l=j+\frac{1}{2}$ and $l=j-\frac{1}{2}$. By introducing the index k, the final form of the kinetic energy reads

$$\begin{aligned}\boldsymbol{K}^j=&\sum_{k}\sum_{\lambda cf}\widetilde{k}^j_{kk}\Big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\\ &+\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\Big]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k}\sum_{\lambda cf}\widetilde{k}^{j}_{kk}\big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\\
&+\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big]\\
&+\sum_{k}\sum_{\lambda cf}m_{0,k,j+\frac{1}{2}}\big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\\
&-\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big]\\
&+\sum_{k}\sum_{\lambda cf}m_{0,k,j-\frac{1}{2}}\big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\\
&-\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf}\big].
\end{aligned}
\tag{36}
$$

The number operators of quarks and antiquarks (n_q and $n_{\bar{q}}$) are invariant under the unitary transformation described above. They just transform into the new operators that also represent number operators of quarks or antiquarks, the latter represented by holes in the lower levels. The same is true for the mass term—it does not change under this transformation.

Up to this point, however, fermions with the orbital angular momentum $l = j \pm \frac{1}{2}$ are still mixed in by the kinetic energy. The final diagonalization will be achieved by applying a BCS transformation [8]. The transformation is given by

$$
\begin{aligned}
\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf} &= c_{j-\frac{1}{2},k}\boldsymbol{b}^{(k,j)\lambda cf}_{j+\frac{1}{2}} - s_{j-\frac{1}{2},k}\boldsymbol{d}^{\dagger(k,j)\lambda cf}_{j-\frac{1}{2}},\\
\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf} &= s_{j+\frac{1}{2},k}\boldsymbol{b}^{(k,j)\lambda cf}_{j-\frac{1}{2}} + c_{j+\frac{1}{2},k}d^{\dagger(k,j)\lambda cf}_{j+\frac{1}{2}},\\
\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf} &= c_{j-\frac{1}{2},k}\boldsymbol{b}^{\dagger}_{j+\frac{1}{2}(k,j)\lambda cf} - s_{j-\frac{1}{2},k}\boldsymbol{d}_{j-\frac{1}{2}(k,j)\lambda cf},\\
\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})j\lambda cf} &= s_{j+\frac{1}{2},k}\boldsymbol{b}^{\dagger}_{j-\frac{1}{2}(k,j)\lambda cf} + c_{j+\frac{1}{2},k}\boldsymbol{d}_{j+\frac{1}{2}(k,j)\lambda cf},\\
\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf} &= c_{j+\frac{1}{2},k}b^{(k,j)\lambda cf}_{j-\frac{1}{2}} - s_{j+\frac{1}{2},k}\boldsymbol{d}^{\dagger(k,j)\lambda cf}_{j+\frac{1}{2}},\\
\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf} &= s_{j-\frac{1}{2},k}\boldsymbol{b}^{(k,j)\lambda cf}_{j+\frac{1}{2}} + c_{j-\frac{1}{2},k}\boldsymbol{d}^{\dagger(k,j)\lambda cf}_{j-\frac{1}{2}},\\
\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf} &= c_{j+\frac{1}{2},k}\boldsymbol{b}^{\dagger}_{j-\frac{1}{2}(k,j)\lambda cf} - s_{j+\frac{1}{2},k}\boldsymbol{d}_{j+\frac{1}{2}(k,j)\lambda cf},\\
\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j-\frac{1}{2},\frac{1}{2})j\lambda cf} &= s_{j-\frac{1}{2},k}\boldsymbol{b}^{\dagger}_{j+\frac{1}{2}(k,j)\lambda cf} + c_{j-\frac{1}{2},k}\boldsymbol{d}_{j-\frac{1}{2}(k,j)\lambda cf}.
\end{aligned}
\tag{37}
$$

with $c \equiv \cos\theta$ and $s \equiv \sin\theta$, where θ is the Bogolubov angle. Applying this rotation to the kinetic-energy term and including the mass term leads to (from now on we skip the upper index j in k^{j}_{kk} because, from the context, it is clear that we work in a fixed column denoted by j)

$$
\begin{aligned}
K^{j}_{\mathrm{BCS}} &= \int d\boldsymbol{x}\boldsymbol{\psi}^{\dagger}(\boldsymbol{x})(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\boldsymbol{\beta}m_0)\boldsymbol{\psi}(\boldsymbol{x})\\
&= \sum_{\lambda cf}\sum_{k}\big\{\big(2\widetilde{k}_{kk}s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}+m_{0,k,j-\frac{1}{2}}c^{2}_{j+\frac{1}{2},k}\\
&\quad -m_{0,k,j+\frac{1}{2}}s^{2}_{j+\frac{1}{2},k}\big)b^{\dagger}_{j-\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j-\frac{1}{2}}\\
&\quad +\big(2\widetilde{k}_{kk}s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}+m_{0,k,j-\frac{1}{2}}c^{2}_{j-\frac{1}{2},k}\\
&\quad -m_{0,k,j+\frac{1}{2}}s^{2}_{j-\frac{1}{2},k}\big)d^{\dagger(kj)\lambda cf}_{j-\frac{1}{2}}d_{j-\frac{1}{2}(kj)\lambda cf}\\
&\quad +\big(2\widetilde{k}_{kk}s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}+m_{0,k,j+\frac{1}{2}}c^{2}_{j-\frac{1}{2},k}\\
&\quad -m_{0,k,j-\frac{1}{2}}s^{2}_{j-\frac{1}{2},k}\big)b^{\dagger}_{j+\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j+\frac{1}{2}}\\
&\quad +\big(2\widetilde{k}_{kk}s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}+m_{0,k,j+\frac{1}{2}}c^{2}_{j+\frac{1}{2},k}\\
&\quad -m_{0,k,j-\frac{1}{2}}s^{2}_{j+\frac{1}{2},k}\big)d^{\dagger(kj)\lambda cf}_{j+\frac{1}{2}}d_{j+\frac{1}{2}(kj)\lambda cf}\\
&\quad -12\big[m_{0,k,j-\frac{1}{2}}\big(c^{2}_{j-\frac{1}{2},k}-s^{2}_{j+\frac{1}{2},k}\big)\\
&\quad +m_{0,k,j+\frac{1}{2}}\big(c^{2}_{j+\frac{1}{2},k}-s^{2}_{j-\frac{1}{2},k}\big)\\
&\quad +2\widetilde{k}_{kk}\big(s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}+s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}\big)\big]\big\}.
\end{aligned}
\tag{38}
$$

The kinetic energy operator is diagonalized, requiring that the terms quadratic in the annihilation and creation operators vanish, which leads to the gap equation that determines the Bogolubov angle:

$$
\begin{aligned}
&\big[\widetilde{k}_{kk}\big(c^{2}_{j-\frac{1}{2},k}-s^{2}_{j-\frac{1}{2},k}\big)-\big(m_{0,k,j-\frac{1}{2}}+m_{0,k,j+\frac{1}{2}}\big)s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}\big]\\
&\quad =0,\\
&\big[\widetilde{k}_{kk}\big(c^{2}_{j+\frac{1}{2},k}-s^{2}_{j+\frac{1}{2},k}\big)-\big(m_{0,k,j-\frac{1}{2}}+m_{0,k,j+\frac{1}{2}}\big)s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}\big]\\
&\quad =0.
\end{aligned}
\tag{39}
$$

The new diagonalized operator for a given column j is finally given by

$$
\begin{aligned}
\boldsymbol{K}^{j}_{\mathrm{BCS}} &= \int d\boldsymbol{x}\boldsymbol{\psi}^{\dagger}(\boldsymbol{x})(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\boldsymbol{\beta}m_0)\boldsymbol{\psi}(\boldsymbol{x})\\
&= \sum_{\lambda cf}\sum_{k}\big\{\big(2\widetilde{k}_{kk}s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}+m_{0,k,j-\frac{1}{2}}c^{2}_{j+\frac{1}{2},k}\\
&\quad -m_{0,k,j+\frac{1}{2}}s^{2}_{j+\frac{1}{2},k}\big)b^{\dagger}_{j-\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j-\frac{1}{2}}\\
&\quad +\big(2\widetilde{k}_{kk}s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}+m_{0,k,j-\frac{1}{2}}c^{2}_{j-\frac{1}{2},k}\\
&\quad -m_{0,k,j+\frac{1}{2}}s^{2}_{j-\frac{1}{2},k}\big)d^{\dagger(kj)\lambda cf}_{j-\frac{1}{2}}d_{j-\frac{1}{2}(kj)\lambda cf}\\
&\quad +\big(2\widetilde{k}_{kk}s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}+m_{0,k,j+\frac{1}{2}}c^{2}_{j-\frac{1}{2},k}\\
&\quad -m_{0,k,j-\frac{1}{2}}s^{2}_{j-\frac{1}{2},k}\big)b^{\dagger}_{j+\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j+\frac{1}{2}}\\
&\quad +\big(2\widetilde{k}_{kk}s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}+m_{0,k,j+\frac{1}{2}}c^{2}_{j+\frac{1}{2},k}\\
&\quad -m_{0,k,j-\frac{1}{2}}s^{2}_{j+\frac{1}{2},k}\big)d^{\dagger(kj)\lambda cf}_{j+\frac{1}{2}}d_{j+\frac{1}{2}(kj)\lambda cf}\\
&\quad -12\big[m_{0,k,j-\frac{1}{2}}\big(c^{2}_{j-\frac{1}{2},k}-s^{2}_{j+\frac{1}{2},k}\big)\\
&\quad +m_{0,k,j+\frac{1}{2}}\big(c^{2}_{j+\frac{1}{2},k}-s^{2}_{j-\frac{1}{2},k}\big)\\
&\quad +2\widetilde{k}_{kk}\big(s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}+s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}\big)\big]\big\}.
\end{aligned}
\tag{40}
$$

Introducing a notation $\boldsymbol{A}\boldsymbol{B}$ to represent the $\sum_{\lambda cf} \boldsymbol{A}_{\lambda cf}\boldsymbol{A}^{\lambda cf}$, it can be rewritten as

$$\boldsymbol{K}^{j}_{\rm BCS} = \sum_{k} \big(\epsilon_{bkj+\frac{1}{2}} \boldsymbol{b}^{\dagger}_{j+\frac{1}{2},k}\boldsymbol{b}_{j+\frac{1}{2},k} + \epsilon_{dkj+\frac{1}{2}} \boldsymbol{d}^{\dagger}_{j+\frac{1}{2},k}\boldsymbol{d}_{j+\frac{1}{2},k} + \epsilon_{bkj-\frac{1}{2}} \boldsymbol{b}^{\dagger}_{j-\frac{1}{2},k}\boldsymbol{b}_{j-\frac{1}{2},k} + \epsilon_{dkj-\frac{1}{2}} \boldsymbol{d}^{\dagger}_{j-\frac{1}{2},k}\boldsymbol{d}_{j-\frac{1}{2},k}\big). \quad (41)$$

This allows us to determine the spectrum of such a model including excited states that are to be associated with hadrons. In particular, meson states are constructed from

$$\big[\boldsymbol{b}^{\dagger}_{j\pm\frac{1}{2},k} \otimes \boldsymbol{d}^{\dagger}_{j\pm\frac{1}{2},k'}\big]^{j(0,0)_C F}_{\mu} \quad \big[\boldsymbol{b}^{\dagger}_{j\pm\frac{1}{2},k} \otimes \boldsymbol{d}^{\dagger}_{j\mp\frac{1}{2},k'}\big]^{j(0,0)_C F}_{\mu}, \quad (42)$$

where the j refers to the total spin, $(0,0)_C$ indicates the overall color singlet, and F gives the total meson flavor. The quantity μ is a short-hand notation for all magnetic quantum numbers. The colored states are shifted to larger energies due to the second-order color operator. The structure of the meson spectrum is already quite involved. Examining (42) and applying to these states the energy operator of Eq. (41), we obtain the meson spectrum. In Eq. (41), no explicit dependence on the coupling to spin and flavor appears—only on the new orbital indices k and k', so states with the same spin and flavor are degenerate in energy, given fixed k and k'. Only when the orbital indices k or k' change does the energy changes. For example, we can expect that the lowest states ($k = k' = 1$) with flavor-spin 1 and 0 multiplets with $j = 0$ and 1 are degenerate.

For the baryon spectrum, the states that correspond to three quark operators ($\boldsymbol{b}^{\dagger}$ or $\boldsymbol{d}^{\dagger}$) should be coupled to the desired flavor and spin. The nucleon is then described by the three quark operators that correspond to the lowest energy. Also here, the $F = \frac{1}{2}$, $j = \frac{1}{2}$ multiplet, which contains the nucleons, will be degenerate with the $F = \frac{3}{2}$, $j = \frac{3}{2}$ multiplet, which contains the Λ mesons.

Each quark operator is a complicated function of the bare quark operators of Eq. (5). Thus, in terms of the bare operators, hadrons states contain a sea of quark-antiquark pairs in the background.

As it can be seen, the kinetic-energy operator, including the mass term, can be diagonalized exactly. In the next subsection, we apply the procedure to a four-level system and show how the bilinear equations involving the α and β matrix elements can be solved.

IV. A TEST CASE: THE TREATMENT OF FOUR LEVELS

We consider the four-level case with $j = \frac{1}{2}$, including the mass term. We take the first two p- ($N = 1, 3$) and first two s- ($N' = 0, 2$) orbital levels. The coefficients $\tilde{k}^{j}_{kq}$, α_{jNk}, and $\beta_{jN'q}$ satisfy the relationships

$$\tilde{k}^{j}_{kq} = 0, \quad k \neq q, \quad (43)$$

$$\sum_{q} |\beta_{jN'q}|^2 = 1, \quad (44)$$

$$\sum_{k} |\alpha_{jNk}|^2 = 1. \quad (45)$$

where

$$\tilde{k}^{j}_{kq} = \sum_{N=1,3} \sum_{N'=N-1}^{\min(3,N+1)} |k^{j}_{NN'}| \alpha^{*}_{Nk}\beta_{N'q} \quad (46)$$

and the factors $|k^{j}_{NN'}|$ are given by

$$|k^{j}_{NN'} = \sqrt{\gamma}\left(\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} + \sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\right). \quad (47)$$

This expression shows that $N = 1$ connects to $N' = 0, 2$ and $N = 3$ connects to $N' = 2$, which is due to the use of the harmonic-oscillator basis. The scale $\sqrt{\gamma}$ for this calculation is chosen such that the energy of the lowest meson state will be around 300 MeV, leading to $\gamma = 1.29\ \text{fm}^{-1}$. From here on we drop the upper index j in $k^{j}_{NN'}$. The conditions that need to be satisfied are

$$\begin{aligned} \alpha^2_{11} + \alpha^2_{12} = 1, \quad \alpha^2_{12} + \alpha^2_{32} = 1, \\ \beta^2_{01} + \beta^2_{02} = 1, \quad \beta^2_{02} + \beta^2_{22} = 1, \\ k_{10}|\alpha_{11}\beta_{02} + |k_{12}|\alpha_{11}\beta_{22} + |k_{32}|\alpha_{12}\beta_{22} = 0, \\ k_{10}|\alpha_{12}\beta_{01} + |k_{12}|\alpha_{12}\beta_{02} + |k_{32}|\alpha_{32}\beta_{02} = 0, \end{aligned} \quad (48)$$

where the α and β are real. These equations are solved according to Ref. [13]. The numerical solution of these equations shows the existence of one solution of the set (α_{11}, α_{12}, α_{32}, β_{01}, β_{02}, β_{22}). This set is listed in Table I.

In the transformed basis, the kinetic energy, including the mass term, can be written as

$$\begin{aligned} \boldsymbol{K}^{\frac{1}{2}}_{\rm BCS} = \sum_{\lambda cf}\sum_{k} \big\{ & \big(2\widetilde{k}_{kk}s_{P,k}c_{P,k} + m_{0,k,S}c^2_{P,k} \\ & - m_{0,k,P}s^2_{P,k}\big)b^{\dagger}_{S(kj)\lambda cf}b^{(kj)\lambda cf}_{S} \\ & + \big(2\widetilde{k}_{kk}s_{S,k}c_{S,k} + m_{0,k,S}c^2_{S,k} \\ & - m_{0,k,P}s^2_{S,k}\big)d^{\dagger(kj)\lambda cf}_{S}d_{S(kj)\lambda cf} \\ & + \big(2\widetilde{k}_{kk}s_{S,k}c_{S,k} + m_{0,k,P}c^2_{S,k} \\ & - m_{0,k,S}s^2_{S,k}\big)b^{\dagger}_{P(kj)\lambda cf}b^{(kj)\lambda cf}_{P} \\ & + \big(2\widetilde{k}_{kk}s_{P,k}c_{P,k} + m_{0,k,P}c^2_{P,k} \end{aligned}$$

TABLE I. Relevant solutions for $t\alpha$ and β for the the system of four levels. The total spin of the level is $j = \frac{1}{2}$ (i.e. the first two s and p orbitals are considered).

Solution	α_{11}	α_{12}	α_{32}	β_{01}	β_{02}	β_{22}	$\widetilde{k}_{11}$ (GeV)	$\widetilde{k}_{22}$ (GeV)
1	−0.707 107	0.707 107	0.707 107	−0.903 453	0.428 687	0.903 453	0.315 08	0.664 03

TABLE II. Solutions of the BCS equation using the α and β values from Table I.

Sol.	θ_1	θ_2	$\epsilon_{S,11}$	$\epsilon_{P,11}$	$\epsilon_{0,11}$	$\epsilon_{S,22}$	$\epsilon_{P,22}$	$\epsilon_{0,22}$
1	0.772 71	0.779 37	0.315 18	0.315 18	7.564 42	0.664 08	0.664 08	15.937 97

$$
\begin{aligned}
&- m_{0,k,S}s_{P,k}^2\big)d_P^{\dagger(kj)\lambda cf}d_{P(kj)\lambda cf}\\
&- 12\big[m_{0,k,S}\big(c_{S,k}^2 - s_{P,k}^2\big) + m_{0,k,P}\big(c_{P,k}^2 - s_{S,k}^2\big)\\
&+ 2\widetilde{k}_{kk}\big(s_{S,k}c_{S,k} + s_{P,k}c_{P,k}\big)\big]\\
&+ \big[\widetilde{k}_{kk}\big(c_{P,k}^2 - s_{P,k}^2\big) - \big(m_{0,k,S} + m_{0,k,P}\big)s_{P,k}c_{P,k}\big]\\
&\times \big[b_{S(kj)\lambda cf}^{\dagger}d_P^{\dagger(kj)\lambda cf} + d_{P(kj)\lambda cf}b_S^{(kj)\lambda cf}\big]\\
&+ \big[\widetilde{k}_{kk}\big(c_{S,k}^2 - s_{S,k}^2\big) - \big(m_{0,k,S} + m_{0,k,P}\big)s_{S,k}c_{S,k}\big]\\
&\times \big[b_{P(kj)\lambda cf}^{\dagger}d_S^{\dagger(kj)\lambda cf} + d_{S(kj)\lambda cf}b_P^{(kj)\lambda cf}\big]\big\}. \qquad (49)
\end{aligned}
$$

Following the procedure described in Sec. III C, we have to solve the gap equations for the mixing angles $\theta_{S,k}$ and $\theta_{P,k}$,

$$
\begin{aligned}
\big[\widetilde{k}_{kk}\big(c_{S,k}^2 - s_{S,k}^2\big) - \big(m_{0,k,S} + m_{0,k,P}\big)s_{S,k}c_{S,k}\big] = 0\\
\big[\widetilde{k}_{kk}\big(c_{P,k}^2 - s_{P,k}^2\big) - (m_{0,k,S} + m_{0,k,P})s_{P,k}c_{P,k}\big] = 0,
\end{aligned} \qquad (50)
$$

respectively, and we find immediately that these equations imply $\theta_{S,k} = \theta_{P,k} = \theta_k$. The θ_k values are listed in Table II for the solution given in Table I. The masses are given by $m_{0,k,S} = \sum_{N'} m_0|\beta_{N'k}|^2$, $m_{0,k,P} = \sum_N m_0|\alpha_{Nk}|^2$. The bare mass $m_0 = 0.008$ GeV is chosen at the intermediate value between the mass of the up and down quarks [14]. Finally, with the values $\widetilde{k}_{11}$ and $\widetilde{k}_{22}$ obtained (see Table I), we can calculate the pair energies $\epsilon_{S,kk}$, $\epsilon_{P,kk}$, and the ground state energy $\epsilon_{0,kk}$:

$$
\begin{aligned}
\big(2\widetilde{k}_{kk}s_{P,k}c_{P,k} + m_{0,S}c_{P,k}^2 - m_{0,P}s_{P,k}^2\big) &= \epsilon_{S,kk}\\
\big(2\widetilde{k}_{kk}s_{S,k}c_{S,k} + m_{0,P}c_{S,k}^2 - m_{0,S}s_{S,k}^2\big) &= \epsilon_{P,kk}\\
12\big[m_{0,S}\big(c_{S,k}^2 - s_{P,k}^2\big) + m_{0,P}\big(c_{P,k}^2 - s_{S,k}^2\big)&\\
+ 2\widetilde{k}_{kk}\big(s_{S,k}c_{S,k} + s_{P,k}c_{P,k}\big)\big] &= \epsilon_{0,kk}.
\end{aligned} \qquad (51)
$$

The results are shown in the Table II.

Taking the solution listed in Table II, we obtain for the kinetic energy of the Hamiltonian

$$
\begin{aligned}
\boldsymbol{K}_{\mathrm{BCS}}^{\frac{1}{2}} = \sum_k \big\{&\epsilon_{S,kk}\big[\boldsymbol{b}_{S(k\frac{1}{2})}^{\dagger}\boldsymbol{b}^{S(k\frac{1}{2})} + \boldsymbol{d}_{S(k\frac{1}{2})}^{\dagger}\boldsymbol{d}^{S(k\frac{1}{2})}\big]\\
&+ \epsilon_{P,kk}\big[\boldsymbol{b}_{P(k\frac{1}{2})}^{\dagger}\boldsymbol{b}^{P(k\frac{1}{2})} + \boldsymbol{d}_{P(k\frac{1}{2})}^{\dagger}\boldsymbol{d}^{P(k\frac{1}{2})}\big]\big\}. \qquad (52)
\end{aligned}
$$

The energies of the lowest s and p orbitals are degenerate, which is due to the structure of the kinetic energy. A residual interaction in the potential term should remove it. Single-particle states belong to the spin-$\frac{1}{2}$, color-$(1, 0)$ and flavor-spin-$\frac{1}{2}$ representations. The antiparticles belong to the conjugate representations. In order to build the low-lying meson states, we have to put one particle and one antiparticle in the lowest level, with energy 0.315 GeV. Thus, the energy of the lightest mesons would be expected around 0.630 GeV, in good agreement with the scale of excitations. Considering that a baryon needs three quarks in order to be in a colorless state, one obtains the first states around 0.945 GeV, which is about the value expected. Note that we used only one parameter ($\sqrt{\gamma}$) in order to adjust the scale of the hadron spectrum! The parameter V_0 does not appear at all because it serves only to shift the colored states to high energy. This feature compares well to other phenomenological models that have more parameters. That the state-splitting is not reproduced is due to the fact that the residual interactions are not yet included within our model. These interactions are well-defined [5] and depend only on the strong coupling constant.

As shown in Sec. II A, the potential term is proportional to the second-order Casimir operator of SU(3) color. This term commutes with the kinetic energy and, thus, one can diagonalize the kinetic energy independently of the potential term.

V. CONCLUSIONS

In this work we have analyzed a QCD-inspired Hamiltonian, considering a constant interaction between quarks. The constant interaction represents an average over all residual interactions. All orbital levels were taken into account in the calculations. The color part was described by an SU(3) group and the flavor part was described by an SU(2) group (only up and down quarks have been included in the model). Nevertheless, it can be easily extended to SU(3) flavor by adding an additional mass term for strange quarks to the kinetic energy. No gluons have been included dynamically. The inclusion of gluons is not a conceptional problem. It will be straight forward, although some work is still required, and will also make use of many-body techniques and group theory. We leave it for a later publication. In this contribution, the system was confined to a finite volume, with the length of an average harmonic oscillator.

The potential interaction resulted in the color-spin operator [second order Casimir operator of the color-SU(3) group]. This interaction splits the colored states from the colorless ones. It shifts the colored states toward higher energies, by the amount proportional to V_0, which is the strength of the constant interaction.

The kinetic-energy operator could be written in terms of the sum of the $\widetilde{\boldsymbol{K}}_{\pm}^{jN}$ operators, restricted to two- and three-level subsystems and excluding the mass term. We proved that they satisfy an SU(2) algebra by adding a $\boldsymbol{K}_0^{jN}$ operator that counts the difference between quarks in the upper and lower orbital levels. This suggests that an analytic or quasi-analytic solution may exists also for the general case. This solution was found after introducing an unitary transformation to the

quark creation and annihilation operators, followed by a simple BCS-type of transformation.

The main features of the calculated spectrum are:

(i) Its structure is already quite rich and resembles in average the physical meson and baryon spectrum. In order to obtain a better agreement to the physical spectrum, corrections still have to be applied, such as the deviation of the average potential to the real one and the lowest order quark-gluon interactions.

(ii) The mesons and baryons consist of two and three partons, respectively, which, once transformed into the original quark operators, correspond to a background sea of many quark-antiquarks pairs.

As shown, although QCD is highly nonperturbative, there is still a lot of room to find analytic solutions for specific sectors of the theory, particularly by applying algebraic and nonperturbative transformations based on many-body techniques. As shown in this contribution, the use of hidden symmetry properties of the QCD Hamiltonian can achieve great simplifications.

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APPENDIX A: CONSTRUCTING THE POTENTIAL INTERACTION

The potential term is given by the color-color interaction

$$\boldsymbol{V} = \int d\boldsymbol{x} d\boldsymbol{y} \boldsymbol{\psi}^\dagger(\boldsymbol{x}) T^a \boldsymbol{\psi}(\boldsymbol{x}) V(|\boldsymbol{x}-\boldsymbol{y}|) \boldsymbol{\psi}^\dagger(\boldsymbol{y}) T^a \boldsymbol{\psi}(\boldsymbol{y}), \tag{A1}$$

where the fermion operators $\boldsymbol{\psi}^\dagger$ and $\boldsymbol{\psi}$ have the following structure:

$$\begin{aligned}
\boldsymbol{\psi}^\dagger(\boldsymbol{x}) &= (\boldsymbol{\psi}_1^\dagger(\boldsymbol{x}), \psi_2^\dagger(\boldsymbol{x})) \\
&= \left(\sum_{Nlm\sigma cf} b^\dagger_{\frac{1}{2}Nlm\sigma cf} R^*_{Nl}(\boldsymbol{x}) Y^*_{lm}(\Omega_x) \chi^\dagger_\sigma, \right. \\
&\quad \left. \times \sum_{Nlm\sigma cf} b^\dagger_{-\frac{1}{2}Nlm\sigma cf} R^*_{Nl}(\boldsymbol{x}) Y^*_{lm}(\Omega_x) \chi^\dagger_\sigma \right), \\
\boldsymbol{\psi}(\boldsymbol{x}) &= \begin{pmatrix} \sum_{Nlm\sigma cf} b^{\frac{1}{2}Nlm\sigma cf} R_{Nl}(\boldsymbol{x}) Y_{lm}(\Omega_x) \chi_\sigma \\ \sum_{Nlm\sigma cf} b^{-\frac{1}{2}Nlm\sigma cf} R_{Nl}(\boldsymbol{x}) Y_{lm}(\Omega_x) \chi_\sigma \end{pmatrix}.
\end{aligned} \tag{A2}$$

Thus, the potential term becomes

$$\begin{aligned}
\boldsymbol{V} = \sum_{\alpha_i N_i, l_i m_i j_i \lambda_i \sigma_i f_i c_i LM} & \int d\boldsymbol{x} d\boldsymbol{y} V(|\boldsymbol{x}-\boldsymbol{y}|) \\
& \times \left\{ \left[b^\dagger_{\alpha_1(N_1,l_1,\frac{1}{2})j_1\lambda_1 c_1 f_1} \left\langle l_1 m_1, \tfrac{1}{2}\sigma_1 \middle| j_1 \lambda_1 \right\rangle \right. \right. \\
& \times R^*_{N_1 l_1}(\boldsymbol{x}) Y^*_{l_1 m_1}(\Omega_x) \chi^\dagger_{\sigma_1} (T^a)^{c_1}_{c_2} \\
& \times b^{\alpha_2(N_2,l_2,\frac{1}{2})j_2\lambda_2 c_2 f_2} \left\langle l_2 m_2, \tfrac{1}{2}\sigma_2 \middle| j_2 \lambda_2 \right\rangle \\
& \times R_{N_2 l_2}(\boldsymbol{x}) Y_{l_2 m_2}(\Omega_x) \chi_{\sigma_2} \delta_{f_1 f_2} \delta_{\alpha_1 \alpha_2} \delta_{\sigma_1 \sigma_2} \Big] \\
& \times \left[b^\dagger_{\alpha_3(N_3,l_3,\frac{1}{2})j_3\lambda_3 c_3 f_3} \left\langle l_3 m_3, \tfrac{1}{2}\sigma_3 \middle| j_3 \lambda_3 \right\rangle \right. \\
& \times R^*_{N_3 l_3}(\boldsymbol{y}) Y^*_{l_3 m_3}(\Omega_y) \chi^\dagger_{\sigma_3} (T^a)^{c_3}_{c_4} \\
& \times b^{\alpha_4(N_4,l_4,\frac{1}{2})j_4\lambda_4 c_4 f_4} \left\langle l_4 m_4, \tfrac{1}{2}\sigma_4 \middle| j_4 \lambda_4 \right\rangle \\
& \left. \left. \times R_{N_4 l_4}(\boldsymbol{y}) Y_{l_4 m_4}(\Omega_y) \chi_{\sigma_4} \delta_{f_3 f_4} \delta_{\alpha_3 \alpha_4} \delta_{\sigma_3 \sigma_4} \right] \right\}.
\end{aligned} \tag{A3}$$

The angular part of the double integral is given by the following expression:

$$\begin{aligned}
& \iint d\Omega_x d\Omega_y Y^*_{l_1 m_1}(\Omega_x) Y_{l_2 m_2}(\Omega_x) \\
& \quad \times V(|\boldsymbol{x}-\boldsymbol{y}|) Y^*_{l_3 m_3}(\Omega_y) Y_{l_4 m_4}(\Omega_4),
\end{aligned} \tag{A4}$$

where the residual interaction can be written as

$$\begin{aligned}
V(|\boldsymbol{x}-\boldsymbol{y}|) &= \sum_L A_L P_L \cos\theta, \\
\Rightarrow A_L &= \left(\frac{2L+1}{2} \right) \int_{-1}^{1} d(\cos\theta) P_L(\cos\theta) V(|\boldsymbol{x}-\boldsymbol{y}|), \\
P_L(\cos\theta) &= \left(\frac{4\pi}{2L+1} \right) \sum_M Y^*_{LM}(\Omega_x) Y_{LM}(\Omega_y).
\end{aligned} \tag{A5}$$

The second expression comes from the orthogonality of the Legendre polynomials and the third is a useful relation for the Legendre Polynomials. Therefore, the double angular integral can be separated into two single angular integrals:

$$\begin{aligned}
& \iint d\Omega_x d\Omega_y Y^*_{l_1 m_1}(\Omega_x) Y_{l_2 m_2}(\Omega_x) V(|\boldsymbol{x}-\boldsymbol{y}|) Y^*_{l_3 m_3}(\Omega_y) Y_{l_4 m_4}(\Omega_y) \\
& = \sum_{LM} A_L \left(\frac{4\pi}{2L+1} \right) (-1)^{m_1+M+m_3} \\
& \quad \times \left[\int d\Omega_x Y_{l_1 - m_1(\Omega_x)} Y_{l_2 m_2(\Omega_x)} Y_{L-M}(\Omega_x) \right] \\
& \quad \times \left[\int d\Omega_y Y_{l_3 - m_3(\Omega_y)} Y_{l_4 m_4(\Omega_y)} Y_{LM}(\Omega_y) \right] \\
& = \sum_{LM} A_L \left(\frac{4\pi}{2L+1} \right) (-1)^{m_1+M+m_3} \\
& \quad \times \left[\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi} \right]^{\frac{1}{2}} \\
& \quad \times \left[\frac{(2l_3+1)(2l_4+1)(2L+1)}{4\pi} \right]^{\frac{1}{2}} \\
& \quad \times \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ -m_1 & m_2 & -M \end{pmatrix} \\
& \quad \times \begin{pmatrix} l_3 & l_4 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ -m_3 & m_4 & M \end{pmatrix}.
\end{aligned} \tag{A6}$$

With the above expressions, the potential term takes the form

$$
\begin{aligned}
\boldsymbol{V} = & \sum_{\alpha_i N_i, l_i m_i j_i \lambda_i \sigma_i f_i c_i L M} \left[b^{\dagger}_{\alpha_1 (N_1, l_1, \frac{1}{2}) j_1 \lambda_1 c_1 f_1} (T^a)^{c_1}_{c_2} b^{\alpha_2 (N_2, l_2, \frac{1}{2}) j_2 \lambda_2 c_2 f_2} \chi^{\dagger}_{\sigma_1} \chi_{\sigma_2} \delta_{f_1 f_2} \delta_{\alpha_1 \alpha_2} \delta_{\sigma_1 \sigma_2} \right] \\
& \times \left[b^{\dagger}_{\alpha_3 (N_3, l_3, \frac{1}{2}) j_3 \lambda_3 c_3 f_3} (T^a)^{c_3}_{c_4} b^{\alpha_4 (N_4, l_4, \frac{1}{2}) j_4 \lambda_4 c_4 f_4} \chi^{\dagger}_{\sigma_3} \chi_{\sigma_4} \delta_{f_3 f_4} \delta_{\alpha_3 \alpha_4} \delta_{\sigma_3 \sigma_4} \right] \\
& \times \int |\boldsymbol{x}|^2 d|\boldsymbol{x}| |\boldsymbol{y}|^2 d|\boldsymbol{y}| R^*_{N_1 l_1}(|\boldsymbol{x}|) R_{N_2 l_2}(|\boldsymbol{x}|) R^*_{N_3 l_3}(|\boldsymbol{y}|) R_{N_4 l_4}(|\boldsymbol{y}|) \\
& \times \int_{-1}^{1} d(\cos\theta) P_L(\cos\theta) V(|\boldsymbol{x} - \boldsymbol{y}|) \left(\frac{2L+1}{2} \right) \prod_{i=1}^{4} \sqrt{(2l_i + 1)(2j_i + 1)} \frac{1}{\sqrt{2L+1}} \frac{1}{\sqrt{2L+1}} \\
& \times (-1)^{m_1 + M + m_3 + l_1 - l_2 + l_3 - l_4 + l_1 - \frac{1}{2} - \lambda_1 + l_2 - \frac{1}{2} - \lambda_2 + l_3 - \frac{1}{2} - \lambda_3 + l_4 - \frac{1}{2} - \lambda_4} \langle l_1 0, l_2 0 | L 0 \rangle \langle l_3 0, l_4 0 | L 0 \rangle \\
& \times \begin{pmatrix} l_1 & \frac{1}{2} & j_1 \\ m_1 & \sigma_1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} l_2 & \frac{1}{2} & j_2 \\ m_2 & \sigma_2 & -\lambda_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ -m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_3 & \frac{1}{2} & j3 \\ m_3 & \sigma_3 & -\lambda_3 \end{pmatrix} \begin{pmatrix} l_4 & \frac{1}{2} & j_4 \\ m_4 & \sigma_4 & -\lambda_4 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ -m_3 & m_4 & M \end{pmatrix},
\end{aligned}
\tag{A7}
$$

$$
\begin{aligned}
& \to \sum_{m_1 \sigma_1 m_2} (-1)^{-\sigma_1 - \frac{1}{2} - \sigma_3 - \frac{1}{2} + M - \lambda_2 - \frac{1}{2} - \lambda_4 - \frac{1}{2}} \begin{pmatrix} l_1 & \frac{1}{2} & j_1 \\ m_1 & \sigma_1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} l_2 & \frac{1}{2} & j_2 \\ m_2 & \sigma_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ -m_1 & m_2 & -M \end{pmatrix} \\
& = \sum_{m_1 \sigma_1 m_2} \begin{pmatrix} j_1 & l_1 & \frac{1}{2} \\ -\lambda_1 & m_1 & \sigma_1 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & \frac{1}{2} \\ -m_2 & \lambda_2 & -\sigma_1 \end{pmatrix} \begin{pmatrix} l_2 & l_1 & L \\ m_2 & -m_1 & -M \end{pmatrix} (-1)^{l_2 + l_1 + \frac{1}{2} + m_2 + m_1 - \sigma_1} \\
& \quad \times (-1)^{-\sigma_1 - \frac{1}{2} - \sigma_3 - \frac{1}{2} + M - \lambda_2 - \frac{1}{2} - \lambda_4 - \frac{1}{2} - l_1 - l_2 - \frac{1}{2} - m_2 - m_1 + \sigma_1 + l_2 + l_1 + L} \\
& = \begin{pmatrix} j_1 & j_2 & L \\ -\lambda_1 & \lambda_2 & -M \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & L \\ l_2 & l_1 & \frac{1}{2} \end{Bmatrix} (-1)^{L - \frac{1}{2} - \sigma_3 - \frac{1}{2} - \lambda_2 - \frac{1}{2} - \lambda_4 - 1}.
\end{aligned}
\tag{A8}
$$

Joining the resulting phase with the last three 3-J coefficients of Eq. (A7), the following structure is obtained:

$$
\begin{aligned}
& \Rightarrow \sum_{m_3 \sigma_3 m_4} (-1)^{L - \frac{1}{2} - \sigma_3 - \frac{1}{2} - \lambda_2 - \frac{1}{2} - \lambda_4 - 1} \begin{pmatrix} l_3 & \frac{1}{2} & j_3 \\ m_3 & \sigma_3 & -\lambda_3 \end{pmatrix} \\
& \quad \times \begin{pmatrix} l_4 & \frac{1}{2} & j_4 \\ m_4 & \sigma_3 & -\lambda_4 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ -m_3 & m_4 & M \end{pmatrix} \\
& = \sum_{m_3 \sigma_3 m_4} \begin{pmatrix} j_3 & l_3 & \frac{1}{2} \\ -\lambda_3 & m_3 & \sigma_3 \end{pmatrix} \begin{pmatrix} l_4 & j_4 & \frac{1}{2} \\ -m_4 & \lambda_4 & -\sigma_3 \end{pmatrix} \\
& \quad \times \begin{pmatrix} l_4 & l_3 & L \\ m_4 & -m_3 & M \end{pmatrix} (-1)^{l_3 + \frac{1}{2} + l_4 + m_4 + m_3 - \sigma_3} \\
& \quad \times (-1)^{L - \frac{1}{2} - \sigma_3 - \frac{1}{2} - \lambda_2 - \frac{1}{2} - \lambda_4 - 1 + l_4 + l_3 + L - l_3 - \frac{1}{2} - l_4 - m_4 - m_3 + \sigma_3} \\
& = \begin{pmatrix} j_3 & j_4 & L \\ -\lambda_3 & \lambda_4 & M \end{pmatrix} \begin{Bmatrix} j_3 & j_4 & L \\ l_4 & l_3 & \frac{1}{2} \end{Bmatrix} (-1)^M (-1)^{\frac{1}{2} + \lambda_2 + \frac{1}{2} + \lambda_4},
\end{aligned}
\tag{A9}
$$

where the 3-J coefficients can be written as

$$
\begin{aligned}
& \begin{pmatrix} j_1 & j_2 & L \\ -\lambda_1 & \lambda_2 & -M \end{pmatrix} \\
& \quad = \frac{(-1)^{j_1 - j_2 + M + j_1 + j_2 - L}}{\sqrt{2L+1}} \langle j_1 \lambda_1, j_2 - \lambda_2 | L - M \rangle,
\end{aligned}
$$

$$
\begin{aligned}
& \begin{pmatrix} j_3 & j_4 & L \\ -\lambda_3 & \lambda_4 & \end{pmatrix} \\
& \quad = \frac{(-1)^{j_3 - j_4 - M + j_3 + j_4 - L}}{\sqrt{2L+1}} \langle j_3 \lambda_3, j_4 - \lambda_4 | L M \rangle.
\end{aligned}
\tag{A10}
$$

Another contribution to the general phase comes from lowering the spin and color indices of the annihilation operators, which gives the additional phases

$$
(-1)^{j_2 - \lambda_2 + \chi_2} (-1)^{j_4 - \lambda_4 + \chi_4}. \tag{A11}
$$

With this, the general phase is given by

$$
\begin{aligned}
& (-1)^M (-1)^{\frac{1}{2} + \lambda_2 + \frac{1}{2} + \lambda_4} (-1)^{j_2 - \lambda_2 + \chi_2} (-1)^{j_4 - \lambda_4 + \chi_4} \\
& \quad \times (-1)^{j_1 - j_2 + M + j_1 + j_2 - L} (-1)^{j_3 - j_4 - M + j_3 + j_4 - L} \\
& \quad = (-1)^{\frac{1}{2} + j_2 + \frac{1}{2} + j_4} (-1)^M (-1)^{\chi_2 + \chi_4},
\end{aligned}
\tag{A12}
$$

where $(-1)^{\chi_k}$ are the SU(3) phases of color, as defined in Ref. [9].

Finally, the product of the SU(3) color generators can be written as follows:

$$
\begin{aligned}
& (\boldsymbol{T}_a)^{c_1}_{c_2} (\boldsymbol{T}^a)^{c_3}_{c_4} \\
& \quad = \langle (1,0) c_2, (1,1) a | (1,0) c_1 \rangle_1 (-1)^{\chi_a} \\
& \qquad \times \langle (1,0) c_4, (1,1) \bar{a} | (1,0) c_3 \rangle_1 \langle (1,0) ||| \boldsymbol{T} ||| (1,0) \rangle^2.
\end{aligned}
\tag{A13}
$$

The factor $\langle (1,0) ||| \boldsymbol{T} ||| (1,0) \rangle$ is a triple-reduced matrix element and its value is given by twice the second order Casimir operator of SU(3) [9], which is equal to $\sqrt{8}$. The bar

over an index refers to the conjugate component. The index 1 at the end of an SU(3) Clebsch-Gordan coefficient is the multiplicity of the coupling. Using the symmetry properties of the SU(3) Clebsch-Gordan coefficients [9] and changing the index a by C, we arrive at the expression

$$(\boldsymbol{T}_C)^{c_1}_{c_2}(\boldsymbol{T}^C)^{c_3}_{c_4} = 3(-1)^{\chi_{c_2}+\chi_{c_4}+\chi_C}\langle(1,0)c_1,(0.1)\bar{c}_2|(1,1)C\rangle_1 \times \langle(1,0)c_3,(0.1)\bar{c}_4|(1,1)\bar{C}\rangle_1. \tag{A14}$$

Finally, the potential term is given by

$$\begin{aligned}\boldsymbol{V} = \sum_{N_i,j_i,\lambda_i,l_i,c_i,\alpha,f,\alpha',f',L,M,C} & \Big[b^{\dagger}_{\alpha_1(N_1,l_1,\frac{1}{2})j_1\lambda_1c_1f}b^{\alpha_2,f}_{(N_2,l_2,\frac{1}{2})j_2-\lambda_2\bar{c}_2} \\ & \times\langle j_1\lambda_1,\, j_2-\lambda_2|L-M\rangle\langle(1,0)c_1,(0,1)\bar{c}_2|(1,1)C\rangle\Big] \\ & \times\Big[b^{\dagger}_{\alpha_3(N_3,l_3,\frac{1}{2})j_3\lambda_3c_3f'}b^{\alpha_4,f'}_{(N_4,l_4,\frac{1}{2})j_4-\lambda_4\bar{c}_4}\langle j_3\lambda_3,\, j_4-\lambda_4|LM\rangle \\ & \times\langle(1,0)c_3,(0,1)\bar{c}_4|(1,1)\bar{C}\rangle\Big](-1)^{M+\chi_C} \\ & \times(-1)^{\frac{1}{2}+j_2+\frac{1}{2}+j_4}\left(\frac{3}{2}\right)\frac{1}{2L+1}\int|\boldsymbol{x}|^2d|\boldsymbol{x}||\boldsymbol{y}|^2d|\boldsymbol{y}|R^*_{N_1l_1} \\ & \times(|\boldsymbol{x}|)R_{N_2l_2}(|\boldsymbol{x}|)R^*_{N_3l_3}(|\boldsymbol{y}|)R_{N_4l_4}(|\boldsymbol{y}|) \\ & \times\int_{-1}^{1}d(\cos\theta)P_L(\cos\theta)V(|\boldsymbol{x}-\boldsymbol{y}|) \\ & \times\prod_{i=1}^{4}\sqrt{(2l_i+1)(2j_i+1)}\langle l_10,l_20|L0\rangle\langle l_30,l_40|L0\rangle \\ & \times\begin{Bmatrix} j_1 & l_1 & \frac{1}{2}\\ l_2 & j_2 & L\end{Bmatrix}\begin{Bmatrix} j_3 & l_3 & \frac{1}{2}\\ l_4 & j_4 & L\end{Bmatrix}.\end{aligned} \tag{A15}$$

The Clebsch-Gordan and all recoupling coefficients are calculated numerically [11,12]. In Ref. [9], a useful collection of formulas is given that comprises all symmetry relations known for these coefficients. In Refs. [15,16], more recent versions of the computer routines for calculating the SU(3) Clebsch-Gordan and recoupling coefficients are available.

As mentioned before, the potential term has the following form:

$$\begin{aligned}\boldsymbol{V} = \sum_{N_i,j_i,\lambda_i,l_i,c_i,\alpha,f,\alpha',f',L,M,C} & (-1)^M(-1)^{\chi_C} \\ & \times\Big[\boldsymbol{b}^{\dagger}_{\alpha(N_1,l_1,\frac{1}{2})j_1\lambda_1,c_1f}\boldsymbol{b}^{\alpha,f}_{(N_2,l_2,\frac{1}{2})j_2\lambda_2,c_2}\langle j_1\lambda_1,\, j_2\lambda_2|LM\rangle \\ & \times\langle(1,0)c_1,(0,1)c_2|(1,1)C\rangle\Big] \\ & \times\Big[\boldsymbol{b}^{\dagger}_{\alpha',(N_3,l_3,\frac{1}{2})j_3,\lambda_3,c_3f'}\boldsymbol{b}^{\alpha',f'}_{(N_4,l_4,\frac{1}{2})j_4\lambda_4,c_4}\langle j_3\lambda_3,\, j_4\lambda_4|L-M\rangle \\ & \times\langle(1,0)c_3,(0,1)c_4|(1,1)\bar{C}\rangle\Big] \\ & \times\sqrt{(2j_1+1)(2j_3+1)}V(N_i,l_i,j_i,L).\end{aligned} \tag{A16}$$

The creation operator is in a (1,0) SU(3)-color irrep and the annihilation operator with a lower index is in a (0,1) SU(3)-color irrep. A creation operator is coupled with the annihilation operator to a (1,1) SU(3)-color irrep, denoting the color generator. The quantity $V(N_i,l_i,j_i,L)$ is the intensity of each component of the interaction, and it reads

$$\begin{aligned}V(N_i,l_i,j_i,L) & = \left(\frac{3}{2}\right)\frac{1}{(2L+1)}(-1)^{j_2+\frac{1}{2}+j_4+\frac{1}{2}} \\ & \times\int|\boldsymbol{x}|^2d|\boldsymbol{x}||\boldsymbol{y}|^2d|\boldsymbol{y}|R^*_{N_1l_1}(|\boldsymbol{x}|)R_{N_2l_2}(|\boldsymbol{x}|) \\ & \times R^*_{N_3l_3}(|\boldsymbol{y}|)R_{N_4l_4}(|\boldsymbol{y}|) \\ & \times\int_{-1}^{1}d(\cos\theta)P_L(\cos\theta)V(|\boldsymbol{x}|,|\boldsymbol{y}|,\cos\theta) \\ & \times\frac{\prod_{i=1}^{4}\sqrt{(2l_i+1)(2j_i+1)}\langle l_10,l_20|L0\rangle\langle l_30,l_40|L0\rangle}{\sqrt{(2j_1+1)(2j_3+1)}} \\ & \times\begin{Bmatrix} j_1 & l_1 & \frac{1}{2}\\ l_2 & j_2 & L\end{Bmatrix}\begin{Bmatrix} j_3 & l_3 & \frac{1}{2}\\ l_4 & j_4 & L\end{Bmatrix}.\end{aligned} \tag{A17}$$

By taking into account that the only allowed value of L is $L=0$ and using a constant potential V_0 as residual interaction, the following simplifications are implied:

$$\begin{aligned}&\int_{-1}^{1}d(\cos\theta)P_L(\cos\theta)V(|\boldsymbol{x}|,|\boldsymbol{y}|,\cos\theta) \\ &\quad = 2V_0\int|\boldsymbol{x}|^2d|\boldsymbol{x}||\boldsymbol{y}|^2d|\boldsymbol{y}|R^*_{N_1l_1}(|\boldsymbol{x}|)R_{N_2l_2}(|\boldsymbol{x}|), \\ &R^*_{N_3l_3}(|\boldsymbol{y}|)R_{N_4l_4}(|\boldsymbol{y}|) = \int|\boldsymbol{x}|^2d|\boldsymbol{x}|R^*_{N_1l_1}(|\boldsymbol{x}|)R_{N_2l_2}(|\boldsymbol{x}|), \\ &\int|\boldsymbol{y}|^2d|\boldsymbol{y}|R^*_{N_3l_3}(|\boldsymbol{y}|)R_{N_4l_4}(|\boldsymbol{y}|) = \delta_{N_1N_2}\delta_{N_3N_4}\delta_{l_1l_2}\delta_{l_3l_4}, \\ &\langle l_10,l_20|L0\rangle = \frac{(-1)^{l_1}}{\sqrt{2l_1+1}}, \\ &\langle l_30,l_40|L0\rangle = \frac{(-1)^{l_3}}{\sqrt{2l_3+1}}, \\ &\begin{Bmatrix} j_1 & j_2 & 0\\ l_1 & l_2 & \frac{1}{2}\end{Bmatrix} = \frac{(-1)^{j_1+l_1+\frac{1}{2}}}{\sqrt{(2j_1+1)(2l_1+1)}}, \\ &\begin{Bmatrix} j_3 & j_4 & 0\\ l_3 & l_4 & \frac{1}{2}\end{Bmatrix} = \frac{(-1)^{j_3+l_3+\frac{1}{2}}}{\sqrt{(2j_3+1)(2l_3+1)}}.\end{aligned}$$

Therefore, the intensities $V(N_i,l_i,j_i,L)$ are given by

$$V(N_i,l_i,j_i,L) = \frac{V_0}{2}\delta_{L0}\delta_{l_1l_2}\delta_{l_3l_4}\delta_{j_1j_2}\delta_{j_3j_4}\delta_{N_1N_2}\delta_{N_3N_4}. \tag{A18}$$

APPENDIX B: CONSTRUCTING THE KINETIC ENERGY OPERATOR

We start from the expression of the kinetic-energy term obtained in Sec. II, but without the mass term. The contribution of the mass term will be considered Appendix C.

$$\widetilde{\boldsymbol{K}}_j = \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} k_{jNN'} \left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} + \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \right]$$
$$+ \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} k^{*}_{jNN'} \left[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} + \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \right] \tag{B1}$$

where N varies in steps of 2 and the $k_{jNN'}$ constant is given by

$$k_{jNN'} = i \int r^2 dr \left[R^{*}_{N(j+\frac{1}{2})}(r) \left(\frac{d}{dr} - \frac{j-\frac{1}{2}}{r} \right) R_{N'(j-\frac{1}{2})}(r) \right]. \tag{B2}$$

We use the three-dimensional harmonic oscillator (i.e., $R_{n'l}(r) = N_{n'l} \exp(-\frac{\gamma r^2}{2}) r^l L^{l+\frac{1}{2}}_{n'}(\gamma r^2)]$ as radial functions and the convention $N' = \frac{n'-l}{2}$, which is the conventional notation for the harmonic oscillator (at the end we will come back to the Nj notation, where $l = j - \frac{1}{2}$ and $l' = j + \frac{1}{2}$). The factors $k_{jNN'}$ can be written now as

$$K_{lnn'} = i \int r^2 dr \left[R^{*}_{nl'}(r) \left(\frac{d}{dr} - \frac{l}{r} \right) R_{n'l}(r) \right]. \tag{B3}$$

The result of applying of the operator $(\frac{d}{dr} - \frac{l}{r})$ on the radial function is

$$\left(\frac{d}{dr} - \frac{l}{r} \right) R_{n'l}(r) = N_{n'l} \left[r^l \exp\left(-\frac{\gamma r^2}{2} \right) \frac{d}{dr} L^{l+\frac{1}{2}}_{n'}(\gamma r^2) - \gamma r^{l+1} \exp\left(-\frac{\gamma r^2}{2} \right) L^{l+\frac{1}{2}}_{n'}(\gamma r^2) \right]. \tag{B4}$$

From the recurrence relations of the Laguerre polynomials

$$L^{\alpha}_{n}(x) = L^{\alpha+1}_{n}(x) - L^{\alpha+1}_{n-1}(x), \quad \frac{d}{dx} L^{\alpha}_{n} = -L^{\alpha+1}_{n-1}(x), \tag{B5}$$

the first relation of Eq. (B4) can be written as

$$\left(\frac{d}{dr} - \frac{l}{r} \right) R_{n'l}(r) = N_{n'l} \left\{ r^l \exp\left(-\frac{\gamma r^2}{2} \right) \frac{d}{dr} L^{l+\frac{1}{2}}_{n'}(\gamma r^2) - \gamma r^{l+1} \exp\left(-\frac{\gamma r^2}{2} \right) \times \left[L^{l+1+\frac{1}{2}}_{n'}(\gamma r^2) - L^{l+1+\frac{1}{2}}_{n'-1}(\gamma r^2) \right] \right\} \tag{B6}$$

and

$$r^l \exp\left(-\frac{\gamma r^2}{2} \right) \frac{d}{dr} L^{l+\frac{1}{2}}_{n'}(\gamma r^2) = r^l \exp\left(-\frac{\gamma r^2}{2} \right) (-2\gamma r) L^{l+1+\frac{1}{2}}_{n'-1}(x). \tag{B7}$$

We obtain the final expression

$$\begin{aligned} \left(\frac{d}{dr} - \frac{l}{r} \right) R_{n'l}(r) &= N_{n'l} \left[-2\gamma r^{l+1} \exp\left(-\frac{\gamma r^2}{2} \right) L^{l+1+\frac{1}{2}}_{n'-1}(\gamma r^2) \right. \\ &\quad - \gamma r^{l+1} \exp\left(-\frac{\gamma r^2}{2} \right) L^{l+1+\frac{1}{2}}_{n'}(\gamma r^2) \\ &\quad \left. + \gamma r^{l+1} \exp\left(-\frac{\gamma r^2}{2} \right) L^{l+1+\frac{1}{2}}_{n'-1}(\gamma r^2) \right] \\ &= N_{n'l} \left[-\gamma \frac{R_{n'-1,l+1}(r)}{N_{n'-1,l+1}} - \gamma \frac{R_{n',l+1}(r)}{N_{n',l+1}} \right]. \end{aligned} \tag{B8}$$

Therefore, the connections for the harmonic oscillator are given by

$$k_{lnn'} = -i\gamma \frac{N_{n',l}}{N_{n'-1,l+1}} \delta_{n,n'-1} \delta_{l',l+1} - i\gamma \frac{N_{n',l}}{N_{n',l+1}} \delta_{n,n'} \delta_{l',l+1}. \tag{B9}$$

In order to translate this expressions into the original notation, we write $n' = \frac{N'-l}{2}$ and $n = \frac{N-l'}{2}$, so for the first and second terms of Eq. (B9) one has the following selection rules:

$$n = n' - 1 \Rightarrow N' = N + 1, \quad n = n' \Rightarrow N' = N - 1. \tag{B10}$$

The quotient to the normalization constants is given by

$$\begin{aligned} \frac{N_{n',l}}{N_{n'-1,l+1}} \delta_{n,n'-1} \delta_{l',l+1} &= \left[\frac{2(n')!}{\Gamma\left(n'+l+\frac{3}{2}\right)} \right]^{\frac{1}{2}} \gamma^{\frac{3}{4}+\frac{l}{2}} \left[\frac{\Gamma\left(n'-1+l+1+\frac{3}{2}\right)}{2(n'-1)!} \right]^{\frac{1}{2}} \\ &\quad \times \frac{1}{\gamma^{\frac{3}{4}+\frac{l+1}{2}}} \\ &= \sqrt{\frac{N-j+\frac{3}{2}}{2\gamma}} \delta_{N',N+1} \end{aligned} \tag{B11}$$

and

$$\begin{aligned} \frac{N_{n',l}}{N_{n',l+1}} \delta_{n,n'} \delta_{l',l+1} &= \left[\frac{2(n')!}{\Gamma(n'+l+\frac{3}{2})} \right]^{\frac{1}{2}} \gamma^{\frac{3}{4}+\frac{l}{2}} \left[\frac{\Gamma\left(n'+l+1+\frac{3}{2}\right)}{2(n')!} \right]^{\frac{1}{2}} \frac{1}{\gamma^{\frac{3}{4}+\frac{l+1}{2}}} \\ &= \sqrt{\frac{N+j+\frac{3}{2}}{2\gamma}} \delta_{N',N-1}. \end{aligned} \tag{B12}$$

The kinetic-energy term is written as $\boldsymbol{K} = \boldsymbol{K}^{(1+2)} + \boldsymbol{K}^{(3+4)}$, with

$$\begin{aligned}\boldsymbol{K}^{(1+2)} = \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} \Bigg[& -i\sqrt{\gamma}\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} \\ & - i\sqrt{\gamma}\sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\Bigg] \\ & \times \big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\big], \end{aligned} \tag{B13}$$

$$\begin{aligned}\boldsymbol{K}^{(3+4)} = \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} \Bigg[& i\sqrt{\gamma}\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} \\ & + i\sqrt{\gamma}\sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\Bigg] \\ & \times \big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big]. \end{aligned} \tag{B14}$$

Finally, the operators $\widetilde{\boldsymbol{K}}_+ = K^{(1)} + K^{(3)}$ and $\widetilde{\boldsymbol{K}}_- = K^{(2)} + K^{(4)}$ have the structure

$$\begin{aligned}\widetilde{\boldsymbol{K}}_+ = (-i\sqrt{\gamma}) & \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N+1}^{N-1} \sum_{\lambda cf} \\ & \times \left(\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} + \sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\right) \\ & \times \big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & - \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big], \end{aligned}$$

$$\begin{aligned}\widetilde{\boldsymbol{K}}_- = (-i\sqrt{\gamma}) & \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} \\ & \times \left(\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} + \sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\right) \\ & \times \big[\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & - \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big]. \end{aligned} \tag{B15}$$

Now, for each set N, j with $l = j + \frac{1}{2}$, we redefine the fermion creation operators by multiplying them by $-i$ and the corresponding annihilation operator by $+i$. In this way, the anticommutation relations are preserved. Explicitly, the mapping is given by

$$\begin{aligned}(-i)\boldsymbol{b}^{\dagger}_{\pm\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} &\to \boldsymbol{b}^{\dagger}_{\pm\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}, \\ (i)\boldsymbol{b}^{\pm\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf} &\to \boldsymbol{b}^{\pm\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}. \end{aligned} \tag{B16}$$

With this the kinetic energy parts acquire the form

$$\begin{aligned}\widetilde{\boldsymbol{K}}_+ = (\sqrt{\gamma}) & \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} \\ & \times \left(\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} + \sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\right) \\ & \times \big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big], \end{aligned}$$

$$\begin{aligned}\widetilde{\boldsymbol{K}}_- = (\sqrt{\gamma}) & \sum_j \sum_{N=j+\frac{1}{2}}^{\infty} \sum_{N'=N-1}^{N+1} \sum_{\lambda cf} \\ & \times \left(\sqrt{\frac{N-j+\frac{3}{2}}{2}}\delta_{N',N+1} + \sqrt{\frac{N+j+\frac{3}{2}}{2}}\delta_{N',N-1}\right) \\ & \times \big[\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf} \\ & + \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})j\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})j\lambda cf}\big]. \end{aligned} \tag{B17}$$

APPENDIX C: CONTRIBUTION OF THE MASS TERM

Substituting the fermion fields into the expression of the mass term, we obtain

$$\begin{aligned}&\int d\boldsymbol{x}\,\boldsymbol{\psi}^{\dagger}(\boldsymbol{x})\beta m_0\boldsymbol{\psi}(\boldsymbol{x}) \\ &= \int d\boldsymbol{x}(\boldsymbol{\psi}_1^{\dagger}(\boldsymbol{x}), \boldsymbol{\psi}_2^{\dagger}(\boldsymbol{x}))\begin{pmatrix} m_0\mathbf{1} & 0 \\ 0 & -m_0\mathbf{1}\end{pmatrix}\begin{pmatrix}\boldsymbol{\psi}_1(\boldsymbol{x}) \\ \boldsymbol{\psi}_2(\boldsymbol{x})\end{pmatrix} \\ &= m_0\int d\boldsymbol{x}(\boldsymbol{\psi}_1^{\dagger}(\boldsymbol{x})\boldsymbol{\psi}_1(\boldsymbol{x}) - \boldsymbol{\psi}_2^{\dagger}(\boldsymbol{x})\boldsymbol{\psi}_2(\boldsymbol{x})) \\ &= m_0\int d\boldsymbol{x} \sum_{N_1N_3l_1l_3m_1m_3j_1j_3\lambda_1\lambda_3\sigma_1\sigma_3cf} \left\langle l_1m_1, \frac{1}{2}\sigma_1\middle| j_1\lambda_1\right\rangle \\ &\quad \times \left\langle l_3m_3, \frac{1}{2}\sigma_3\middle| j_3\lambda_3\right\rangle R^*_{N_1l_1}(|\boldsymbol{x}|)R_{N_3l_3}(|\boldsymbol{x}|)Y^*_{l_1m_1}(\Omega_x) \\ &\quad \times Y_{l_3m_3}(\Omega_x)\chi^*_{\sigma_1}\chi_{\sigma_3}\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N_1l_1\frac{1}{2})j_1\lambda_1cf}\boldsymbol{b}^{\frac{1}{2}(N_3l_3\frac{1}{2})j_3\lambda_3cf} \\ &\quad - \int d\boldsymbol{x} \sum_{N_2N_4l_2l_4m_2m_4j_2j_4\lambda_2\lambda_4\sigma_2\sigma_4cf} \left\langle l_2m_2, \frac{1}{2}\sigma_2\middle| j_2\lambda_2\right\rangle \\ &\quad \times \left\langle l_4m_4, \frac{1}{2}\sigma_4\middle| j_4\lambda_4\right\rangle R^*_{N_2l_2}(|\boldsymbol{x}|)R_{N_4l_4}(|\boldsymbol{x}|)Y^*_{l_2m_2}(\Omega_x)Y_{l_4m_4} \\ &\quad \times (\Omega_x)\chi^*_{\sigma_2}\chi_{\sigma_4}\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N_2l_2\frac{1}{2})j_2\lambda_2cf}\boldsymbol{b}^{-\frac{1}{2}(N_4l_4\frac{1}{2})j_4\lambda_4cf}. \end{aligned} \tag{C1}$$

The radial and the angular integral leads to the restriction $\delta_{N_1N_3}\delta_{l_1l_3}\delta_{m_1m_3}\delta_{\sigma_1\sigma_3}$ and $\delta_{N_2N_4}\delta_{l_2l_4}\delta_{m_2m_4}\delta_{\sigma_2\sigma_4}$, respectively. Thus, the mass term can be written as [applying also the mapping of the fermion creation and annihilation operators, as given in Eq. (B16)]

$$\begin{aligned}
&\int d\boldsymbol{x}\boldsymbol{\psi}^{\dagger}(\boldsymbol{x})\beta m_0\boldsymbol{\psi}(\boldsymbol{x})\\
&= m_0\sum_{Nlj\lambda cf}\big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(Nl\frac{1}{2})jj\lambda cf}\boldsymbol{b}^{\frac{1}{2}(Nl\frac{1}{2})jj\lambda cf}\\
&\quad-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}N(l\frac{1}{2})jj\lambda cf}\boldsymbol{b}^{-\frac{1}{2}N(l\frac{1}{2})jj\lambda cf}\big]\\
&= m_0\sum_{j}\sum_{\lambda cf}\sum_{N'=j-\frac{1}{2}}^{\infty}\big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N',j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\big]\\
&\quad+m_0\sum_{j}\sum_{\lambda cf}\sum_{N=j+\frac{1}{2}}^{\infty}\big[\boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\boldsymbol{b}^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad-\boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\big],
\end{aligned}\tag{C2}$$

where N and N' vary in steps of 2. Now, we apply the same unitary transformation as described in the text; that is,

$$\begin{aligned}
&\int d\boldsymbol{x}\boldsymbol{\psi}^{\dagger}(\boldsymbol{x})\beta m_0\boldsymbol{\psi}(\boldsymbol{x})\\
&=\sum_{j}\sum_{\lambda cf}\sum_{q}\big\{m_{0,q,j-\frac{1}{2}}\big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad-\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(q,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\big]\big\}\\
&\quad+\sum_{j}\sum_{\lambda cf}\sum_{k}\big\{m_{0,k,j+\frac{1}{2}}\big[\widehat{\boldsymbol{b}}^{\dagger}_{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\widehat{\boldsymbol{b}}^{\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad-\widehat{\boldsymbol{b}}^{\dagger}_{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\widehat{\boldsymbol{b}}^{-\frac{1}{2}(k,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\big]\big\}.
\end{aligned}\tag{C3}$$

By applying the method described in Sec. III, the only remaining terms correspond to $k=q$. With this, the masses become

$$\begin{aligned}
m_{0,k,j-\frac{1}{2}} &= \sum_{N'=j-\frac{1}{2}}^{n-1} m_0|\beta_{jN'k}|^2,\\
m_{0,k,j+\frac{1}{2}} &= \sum_{N=j+\frac{1}{2}}^{n} m_0|\alpha_{jNk}|^2.
\end{aligned}\tag{C4}$$

Finally, the mass contribution to the BCS kinetic energy is given by

$$\begin{aligned}
&\boldsymbol{K}^{\mathrm{BCS}}_{\mathrm{mass}}\\
&=\sum_{\lambda cf}\sum_{kj}\big\{\big(m_{0,k,j-\frac{1}{2}}c^2_{j+\frac{1}{2},k}-m_{0,k,j+\frac{1}{2}}s^2_{j+\frac{1}{2},k}\big)b^{\dagger}_{j-\frac{1}{2}(kj)\lambda cf}\\
&\quad\times b^{(kj)\lambda cf}_{j-\frac{1}{2}}+\big(m_{0,k,j-\frac{1}{2}}c^2_{j-\frac{1}{2},k}-m_{0,k,j+\frac{1}{2}}s^2_{j-\frac{1}{2},k}\big)d^{\dagger(kj)\lambda cf}_{j-\frac{1}{2}}\\
&\quad\times d_{j-\frac{1}{2}(kj)\lambda cf}+\big(m_{0,k,j+\frac{1}{2}}c^2_{j-\frac{1}{2},k}-m_{0,k,j-\frac{1}{2}}s^2_{j-\frac{1}{2},k}\big)\\
&\quad\times b^{\dagger}_{j+\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j+\frac{1}{2}}+\big(m_{0,k,j+\frac{1}{2}}c^2_{j+\frac{1}{2},k}-m_{0,k,j-\frac{1}{2}}s^2_{j+\frac{1}{2},k}\big)\\
&\quad\times d^{\dagger(kj)\lambda cf}_{j+\frac{1}{2}}d_{j+\frac{1}{2}(kj)\lambda cf}+12\big[m_{0,k,j-\frac{1}{2}}\big(s^2_{j+\frac{1}{2},k}-c^2_{j-\frac{1}{2},k}\big)\\
&\quad+m_{0,k,j+\frac{1}{2}}\big(s^2_{j-\frac{1}{2},k}-c^2_{j+\frac{1}{2},k}\big)\big]-\big[\big(m_{0,k,j-\frac{1}{2}}+m_{0,k,j+\frac{1}{2}}\big)\\
&\quad\times s_{j+\frac{1}{2},k}c_{j+\frac{1}{2},k}\big]\big[b^{\dagger}_{j-\frac{1}{2}(kj)\lambda cf}d^{\dagger(kj)\lambda cf}_{j+\frac{1}{2}}+d_{j+\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j-\frac{1}{2}}\big]\\
&\quad-\big[\big(m_{0,k,j-\frac{1}{2}}+m_{0,k,j+\frac{1}{2}}\big)s_{j-\frac{1}{2},k}c_{j-\frac{1}{2},k}\big]\\
&\quad\times\big[b^{\dagger}_{j+\frac{1}{2}(kj)\lambda cf}d^{\dagger(kj)\lambda cf}_{j-\frac{1}{2}}+d_{j-\frac{1}{2}(kj)\lambda cf}b^{(kj)\lambda cf}_{j+\frac{1}{2}}\big]\big\}.
\end{aligned}\tag{C5}$$

APPENDIX D: THE KINETIC ENERGY TERM FOR THREE LEVELS AND A FIXED N AND j

The Dirac picture of the three-energy levels implies six states, three with positive energy and three with negative energy. Fixing j and N, we are working in a determined column with total spin j and the possible connections $(N-1, j-\frac{1}{2}) \longleftrightarrow (N, j+\frac{1}{2})$ and $(N, j+\frac{1}{2}) \longleftrightarrow (N+1, j-\frac{1}{2})$. The operators $\boldsymbol{K}_+$ and $\boldsymbol{K}_-$ of this case have the following structure, respectively:

$$\begin{aligned}
\widetilde{\boldsymbol{K}}_+ &= \sqrt{\gamma}\Bigg[\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad+\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\Bigg],
\end{aligned}\tag{D1}$$

$$\begin{aligned}
\widetilde{\boldsymbol{K}}_- &= \sqrt{\gamma}\Bigg[\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad+\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\\
&\quad+\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}b^{\dagger}_{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})jj\lambda cf}b^{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})jj\lambda cf}\Bigg].
\end{aligned}\tag{D2}$$

The commutation relation of these operators is given by (omitting the subindices $j\lambda cf$ for simplicity)

$$
\begin{aligned}
[\widetilde{\boldsymbol{K}}_+, \widetilde{\boldsymbol{K}}_-] = \gamma \Bigg\{ & \left(\frac{N-j+\frac{3}{2}}{2}\right)\left[\boldsymbol{N}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} - \boldsymbol{N}_{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\right] - \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} \\
& \times \boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} - \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} \\
& + \left(\frac{N+j+\frac{3}{2}}{2}\right)\left[\boldsymbol{N}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} - \boldsymbol{N}_{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}\right] + \left(\frac{N-j+\frac{3}{2}}{2}\right)\left[\boldsymbol{N}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} - \boldsymbol{N}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\right] \\
& + \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}}\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \\
& \times \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N+j+\frac{3}{2}}{2}\right)\left[\boldsymbol{N}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} - \boldsymbol{N}_{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\right]\Bigg\} \\
= {} & 2\widetilde{\boldsymbol{K}}_0, \qquad \text{(D3)}
\end{aligned}
$$

$$
\begin{aligned}
[\widetilde{\boldsymbol{K}}_0, \widetilde{\boldsymbol{K}}_+] = {} & \left(\frac{\gamma^{\frac{3}{2}}}{2}\right)\Bigg\{\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} \\
& \times \boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N+j+\frac{3}{2}}{2}\right)\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \\
& \times \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} \\
& + \left(\frac{N+j+\frac{3}{2}}{2}\right)\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} \\
& + \left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} \\
& + \left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} \\
& + \left(\frac{N+j+\frac{3}{2}}{2}\right)\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} + \left(\frac{N-j+\frac{3}{2}}{2}\right)\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})} \\
& \times \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} + \left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} + \left(\frac{N+j+\frac{3}{2}}{2}\right)\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} \\
& \boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})} + \left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{3}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\Bigg\} \\
= {} & \frac{\gamma}{2}\sqrt{\gamma}\Bigg\{\left[2\left(\frac{N-j+\frac{3}{2}}{2}\right) + 2\left(\frac{N+j+\frac{3}{2}}{2}\right)\right]\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})} \\
& + \left[2\left(\frac{N-j+\frac{3}{2}}{2}\right) + 2\left(\frac{N+j+\frac{3}{2}}{2}\right)\right]\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}
\end{aligned}
$$

$$+\left[2\left(\frac{N-j+\frac{3}{2}}{2}\right)+2\left(\frac{N+j+\frac{3}{2}}{2}\right)\right]\left(\frac{N-j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N+1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}$$

$$+\left[2\left(\frac{N-j+\frac{3}{2}}{2}\right)+2\left(\frac{N+j+\frac{3}{2}}{2}\right)\right]\left(\frac{N+j+\frac{3}{2}}{2}\right)^{\frac{1}{2}} \boldsymbol{b}^{\dagger}_{\frac{1}{2}(N-1,j-\frac{1}{2},\frac{1}{2})}\boldsymbol{b}^{-\frac{1}{2}(N,j+\frac{1}{2},\frac{1}{2})}\Bigg\}$$

$$=\frac{\gamma}{2}[2N+3]\widetilde{\boldsymbol{K}}_{+}. \tag{D4}$$

Defining

$$\begin{aligned}\boldsymbol{K}_{\pm}&=\eta\widetilde{\boldsymbol{K}}_{\pm},\\ \boldsymbol{K}_{0}&=\eta^{2}\widetilde{\boldsymbol{K}}_{0},\end{aligned} \tag{D5}$$

we arrive at the relation

$$[\boldsymbol{K}_{0},\boldsymbol{K}_{+}]=\frac{\gamma}{2}(2N+3)\,\eta^{2}\boldsymbol{K}_{+}. \tag{D6}$$

Choosing

$$\eta=\left[\sqrt{\frac{\gamma}{2}(2N+3)}\right]^{-1} \tag{D7}$$

we arrive at the well-known SU(2) algebra. Of course, this has to be repeated for $\boldsymbol{K}_{-}$, which is easily obtained by taking the adjoint of $\boldsymbol{K}_{+}$.

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