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A conjecture on the use of quantum algebras in the treatment of discrete systems

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ABSTRACT

The interactions between atomic spin-states, and between them and an external radiation field, can be described in terms of quantum algebras by a trade-off of bosonic and fermionic degrees of freedom and q-deformed schemes. In this Letter we discuss the use of this concept concerning the calculation of a spin observable, like the spin squeezing.

1. Introduction

Recent developments in quantum optics and in quantum computing have motivated several studies of spin systems and their interactions [1,2]. The interest in these systems is linked to potential applications to the field of quantum devices [3,4]. In previous publications we have addressed some questions concerning the response of spin systems to the interaction with radiation [5–7], as well as the response of spin-interacting arrays, like spin chains [8,9]. As discussed there, one of the main difficulties associated to the calculation of spin-squeezing [10,11] is the large dimensionality of the space of configurations needed to built up the spin density matrix. Among different algebraic methods, quantum groups have been, successfully, applied to the study of spin-chains, for a review see, for instance [12]. Also, the use of quantum algebras to treat fermion and boson systems was presented in [13], in dealing with the Dicke model [14]. In this Letter we conjecture about the equivalence of a Hamiltonian which describes spin–spin and spin–radiation interactions [16,15,17,18], and an effective q-deformed Hamiltonian which contains only spin–spin interactions. The conjecture is based on the replacement of the spin–radiation interaction-term by a linear term which is written as a function of the generators of the $su_q(2)$ or $su_q(3)$ algebras. This replacement leads to a purely fermion Hamiltonian with almost the same spectrum and eigenfunctions of the initial fermion–boson Hamiltonian [19]. Here, we shall extend on this notion by exploring the consequences of our conjecture when applied to a system of atoms with two and three levels interacting with a radiation field. The notion that a class of Hamiltonian, with interaction terms between fermions and bosons, may be reduced to effective forms with simpler interactions was explored in Ref. [20]. Therein, the method of small rotations, which may eliminate sectors of the Hamiltonian which commute with the generators of the rotations, was discussed, particularly, for Hamiltonians describing atomic levels interacting with boson modes. The method of Klimov and Sanchez-Soto [20] is making use of the separation of sectors of a given Hamiltonian by projecting them into the subspaces of adequate integrals of motion. Then, by the proper choice of rotation operators one may transform sectors of the Hamiltonian and replace terms of it by effective ones where the absorption of some degrees of freedom is realized by the rotations. The method is mathematically sound and it offers a valid alternative to the path which we are proposing here, that is to elaborate on the deformation of the algebras as a way to eliminate boson degrees of freedom from the original fermion–boson Hamiltonians. In this Letter we concentrate on this procedure as a conjecture and explore its consequences. Further developments will certainly be devoted to establish the link with the method of Klimov and Sanchez-Soto [21]. The formalism is presented in Section 2, where we introduce the essentials of the proposed mapping and elaborate on the relevant symmetries. In Section 3 we show the results of the calculations which we have performed to support the conjecture. The conclusions are drawn in Section 4.

2. Formalism

In this section we shall present the essentials of the formalism for two leading cases, that is two- and three-level atoms interact-

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ing with radiation. As it will be explained below, these two cases involve the deformation of the $su(2)$ and $su(3)$ algebras, respectively, and they have been taken as test-cases.

2.1. Two-level atoms

We shall consider a system of N atoms, each of them having two spin-states and interacting with photons. A physical realization of this system would be the excitation of two-level atoms in a cavity by an incoming photon [18,22].

We write the Hamiltonian of the system as

$$\begin{aligned} H &= \omega_f S_z + \lambda \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \left(S_+^{(j)} S_-^{(i)} + S_+^{(i)} S_-^{(j)}\right) \\ &\quad + \omega_b\left(a^\dagger a + \frac{1}{2}\right) + \eta\left(a^\dagger S_- + S_+ a\right) \\ &\approx \lambda\frac{N}{2}(N-1) + (\omega_f - \lambda)S_z - 2\lambda S_z^2 \\ &\quad + \omega_b\left(a^\dagger a + \frac{1}{2}\right) + \eta\left(a^\dagger S_- + S_+ a\right), \end{aligned} \tag{1}$$

where

$$\begin{aligned} S_+ &= \sum_{j=1}^{N} S_+^{(j)}, \\ S_- &= S_+^\dagger, \\ S_z &= \sum_{j=1}^{N} S_z^{(j)} \end{aligned} \tag{2}$$

are the ladder operators which rise (S_+), or lower (S_-) the states of the atoms, S_z is the number operator; the energy gap between the states of a given atom is ω_f ($\hbar = 1$), and ω_b is the energy of the external boson field (photons). The third term of the Hamiltonian is the free-photon field, and the last term is the interaction of the photons with the atoms. The operators S_+, S_- and S_z obey the commutation rules of the $su(2)$ algebra. The operators $S_\pm^{(j)}$ and $S_z^{(j)}$ are the generators of the j-th copy $su(2)_j$ of the algebra, where j is the atomic index.[1] The excitations of $k \leqslant N$ atoms is described by Dicke states [14]

$$|k\rangle_{at} = \sqrt{\frac{(N-k)!}{N!k!}}(S_+)^k|0\rangle. \tag{3}$$

Since the Hamiltonian of Eq. (1) contains boson degrees of freedom, the state which represents l photons is written as the number state

$$|l\rangle_{ph} = \frac{1}{\sqrt{l!}}a^{\dagger l}|0\rangle. \tag{4}$$

The Hamiltonian of Eq. (1) commutes with the operator

$$\hat{L} = a^\dagger a + S_z + \frac{N}{2}. \tag{5}$$

In terms of $\hat{L}$, the Hamiltonian of Eq. (1) is written as

$$\begin{aligned} H &= \lambda\frac{N}{2}(N-1) + \omega_b\hat{L} - \omega_b\frac{1}{2}(N-1) \\ &\quad + (\omega_f - \omega_b - \lambda)\tilde{S}_z - 2\lambda\tilde{S}_z^2 \\ &\quad + \eta\left(a^\dagger S_- + S_+ a\right). \end{aligned} \tag{6}$$

The basis of Eq. (5) may be labeled by the eigenvalues $\hat{L}$ of the operator L, such that

$$\hat{L}|L,k\rangle = L|L,k\rangle, \tag{7}$$

where $L = l + k$. We shall then express the wave function of the photons and atoms as

$$|L,k\rangle = |L-k\rangle_{ph} \otimes |k\rangle_{at}. \tag{8}$$

In each L-subspace, the eigenvalues and eigenvectors of H, of Eq. (1), can readily be obtained. With the set of these exact solutions of H, the time evolution of the expectation value of a given operator O is written

$$\begin{aligned} \langle O(t)\rangle &= \mathrm{Tr}\left(\rho(t)O\right) \\ &= \sum_{\alpha,\beta}\langle\beta|I\rangle\langle I|\alpha\rangle\langle\alpha|O|\beta\rangle e^{-i(E_\alpha - E_\beta)t}. \end{aligned} \tag{9}$$

In the above equation $\rho(t)$ is the density operator $\rho(t) = U^\dagger(t)\rho(0)U(t)$, being $\rho(0) = |I\rangle\langle I|$; the state $|I\rangle$ is the initial state of the system, $\{E_\alpha\}$ and $\{|\alpha\rangle\}$ are the α-th eigenvalue and eigenvector of the Hamiltonian, and $U(t) = \exp(-iHt)$ is the evolution operator.

2.1.1. The conjecture about an effective $su_q(2)$ Hamiltonian

The quantum algebra $su_q(2)$ is a Hopf algebra deformation of $su(2)$ [23,24] whose generators are $\tilde{S}_\pm$ and $\tilde{S}_z$, which obey the commutation rules

$$[\tilde{S}_z, \tilde{S}_\pm] = \pm\tilde{S}_\pm, \qquad [\tilde{S}_+, \tilde{S}_-] = [2\tilde{S}_z]_q. \tag{10}$$

The q-analogue $[x]_q$ of a given object x (a c-number or an operator) is defined by[2]

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{11}$$

The $su(2)$ algebra is recovered from Eq. (10) in the limit $q \to 1$. When q is not a root of unity, the irreducible representations of $su_q(2)$ are obtained as a straightforward generalization of those of $su(2)$ [24].

To the starting Hamiltonian (1) we assign, by construction, the effective Hamiltonian

$$\begin{aligned} H_{(L,q)} &= \lambda\frac{N}{2}(N-1) + \omega_b L - \omega_b\frac{1}{2}(N-1) \\ &\quad + (\omega_f - \omega_b - \lambda)\tilde{S}_z - 2\lambda\tilde{S}_z^2 \\ &\quad + \chi(L,q)(\tilde{S}_+ + \tilde{S}_-) \end{aligned} \tag{12}$$

where $\chi(L,q)$ is a scalar function

$$\begin{aligned} \chi(L,q) &= \eta\sqrt{\frac{(L-k_m)(k_m+1)(N-k_m)}{[k_m+1]_q[N-k_m]_q}}, \\ k_m &= \frac{1}{3}(-1+L+N) \\ &\quad - \frac{1}{3}\sqrt{1+L+L^2+N-NL+N^2}, \end{aligned} \tag{13}$$

and $H_{(L,q)}$ will be realized in an $su_q(2)$ irreducible representation with the same dimension as the L-subspace (7). Note that, the Hamiltonians H of Eq. (1) and $H_{(L,q)}$ of Eq. (12) are different, since the latter does not have boson degrees of freedom, and in the limit $q \to 1$ $H_{(L,q)} \neq H$. The main result of this procedure is

[1] The tensor product $\prod_{j=1}^{N} su(2)_j$ is the carrier space for the representations of the fermion (atomic) sector of the Hamiltonian.

[2] We shall use q or z ($q = e^z$) as the deformation parameter, and we shall assume that q is real.

that the boson degrees of freedom of Eq. (1) may be absorbed by the q-deformation in Eq. (12), provided that q is defined as an appropriate function of both N and L. This is the trade-off leading to the purely fermionic structure of Eq. (12). In this way it is possible to think of $H_{(L,q)}$ as an effective Hamiltonian with physical properties similar to those of H. In particular, we shall determine, numerically, the optimal values of the deformation parameter q by imposing that the spectrum of the Hamiltonian of Eq. (12) be as close as possible to that of Eq. (1). In so doing, the function $\chi(L,q)$ is fixed to ensure that the maximum values of the interaction terms of the Hamiltonians H and $H_{(L,q)}$ do coincide (see [19]).

2.2. Three-level atoms

The system consists of A identical three-levels atoms interacting with a radiation field [5,16,17]. The atoms and the photons are placed in a cavity.

The creation (annihilation) operator for the i-th atomic level ($i = 0, 1, 2$), is denoted by $b_i^\dagger(b_i)$. The operators $b_i^\dagger$ and b_i obey boson commutation relations.

The Hamiltonian of the system reads

$$H = \omega a^\dagger a + \sum_i E_i S^{ii} + g_1\left(aS_+^{01} + a^\dagger S_-^{01}\right) + g_2\left(aS_+^{12} + a^\dagger S_-^{12}\right). \tag{14}$$

The operators $S_+^{ij}, S_-^{ij}, S_{z\,i\neq j}^{ij}$ generate the $(A+1)(A+2)/2$-dimensional symmetric representation of the $su(3)$ algebra. This can be demonstrated easily, since the operators

$$S^{ij} = b_j^\dagger b_i, \quad i, j = 0, 1, 2, \tag{15}$$

fulfill the commutation relations

$$\left[S^{ij}, S^{km}\right] = \delta_{im}S^{kj} - \delta_{jk}S^{im}. \tag{16}$$

They are the starting operators which are used to define the atomic inversion operators

$$S_z^{ij} = \frac{1}{2}\left(S^{jj} - S^{ii}\right) \tag{17}$$

and the transition operators $S_\pm^{ij}$

$$S_+^{ij} = S^{ij}, \qquad S_-^{ij} = \left(S_+^{ij}\right)^\dagger = S^{ji}, \quad i, j = 0, 1, 2,\ i < j. \tag{18}$$

In the expression of H of Eq. (14), ω is the energy of the photon, $a^\dagger(a)$ is the one photon-creation (-annihilation) operator, E_i is the energy of the i-th atomic level, and g_1 and g_2 are coupling constants describing the absorption (emission) of a photon in the presence of an upward (downward) atomic excitation between levels 0 and 1 (term proportional to g_1), and between levels 1 and 2 (term proportional to g_2).

The two-photon resonance condition [16] is satisfied by fixing the energies of the atomic levels E_i at the values

$$E_2 - E_0 = 2\omega, \qquad E_1 - E_0 = \omega - \Delta. \tag{19}$$

The operator

$$\hat{L} = a^\dagger a + 2S_z^{02} \tag{20}$$

commutes with the Hamiltonian of Eq. (14), which, therefore, can be diagonalized in the basis of states

$$|n_b n_0 n_1 n_2\rangle = N_{n_b n_0 n_1 n_2} a^{\dagger n_b} b_0^{\dagger n_0} b_1^{\dagger n_1} b_2^{\dagger n_2}|0\rangle, \tag{21}$$

and by enforcing the constraints

$$\begin{aligned} &n_0 + n_1 + n_2 = A, \\ &n_2 - n_0 + n_b = L, \end{aligned} \tag{22}$$

where A is the number of atoms, and L is the sum of the number of photons, n_b, and the difference $n_2 - n_0$ between the population of the atomic states $i = 2$ and $i = 0$. $N_{n_b,n_0n_1n_2}$ is a normalization constant.

In terms of the symmetry operator, the Hamiltonian reads

$$H = \omega\hat{L} + (E_0 + \omega)A - \Delta S^{11} + g_1\left(aS_+^{01} + a^\dagger S_-^{01}\right) + g_2\left(aS_+^{12} + a^\dagger S_-^{12}\right). \tag{23}$$

In the basis of states with good L, which is the eigenvalue of $\hat{L}$, the exact solution is written

$$|\Psi_\alpha\rangle = \sum_{a\equiv\{n_b,n_0,n_1,n_2\}} c_\alpha(a)|a\rangle. \tag{24}$$

2.2.1. The conjecture about an effective $su_q(3)$ Hamiltonian

The quantum algebra $su_q(3)$ is a Hopf algebra deformation of $su(3)$ whose generators are [25]

$$\begin{aligned} &\tilde{S}^{01} = \tilde{b}_1^\dagger \tilde{b}_0, \\ &\tilde{S}^{10} = \tilde{b}_0^\dagger \tilde{b}_1, \\ &\tilde{S}^{12} = \tilde{b}_2^\dagger \tilde{b}_1, \\ &\tilde{S}^{21} = \tilde{b}_1^\dagger \tilde{b}_2, \\ &\tilde{S}^{02} = \tilde{b}_2^\dagger \tilde{b}_0 q^{\tilde{N}_1}, \\ &\tilde{S}^{20} = \tilde{b}_0^\dagger \tilde{b}_2 q^{-\tilde{N}_1}, \\ &\tilde{S}^{ii} = N_i, \quad i = 0, 1, 2, \end{aligned} \tag{25}$$

with

$$\begin{aligned} &\left[\tilde{N}_i, \tilde{b}_j^\dagger\right] = \delta_{ij}\tilde{b}_i^\dagger, \\ &\left[\tilde{N}_i, \tilde{b}_j\right] = -\delta_{ij}\tilde{b}_i, \\ &\left[\tilde{b}_i, \tilde{b}_j^\dagger\right]_q = \delta_{ij}q^{-\tilde{N}_i}, \\ &\left[\tilde{b}_i, \tilde{b}_j^\dagger\right]_{q^{-1}} = \delta_{ij}q^{\tilde{N}_i}. \end{aligned} \tag{26}$$

The $su(3)$ algebra is recovered from Eq. (25) in the limit $q \to 1$.

To the starting Hamiltonian (23) we assign, by construction, the effective Hamiltonian

$$H_{eff}(L,q) = \omega L + (E_0 + \omega)A - \Delta\tilde{N}_1 + \chi_1(L,q,g_1)\left(\tilde{S}_+^{01} + \tilde{S}_-^{01}\right) + \chi_2(L,q,g_2)\left(\tilde{S}_+^{12} + \tilde{S}_-^{12}\right), \tag{27}$$

where $\chi_i(L,q,g_i)$ is a scalar function and $H_{eff}(L,q)$ will be realized in an $su_q(3)$ irreducible representation with the same dimension as the L-subspace (21),

$$\chi_1(L,q,g_1) = g_1\sqrt{\frac{(L+n_{0m})(A-n_{0m})n_{0m}}{[n_{0m}]_q[A-n_{0m}]_q}},$$

$$n_{0m} = \frac{1}{3}\left(A - L + \sqrt{L^2 + AL + A^2}\right),$$

$$\chi_2(L,q,g_2) = g_2\sqrt{\frac{(L+n_{0m}-n_{2m})(A-n_{0m}-n_{2m})(n_{2m}+1)}{[n_{2m}+1]_q[A-n_{0m}-n_{2m}]_q}},$$

$$n_{0m} = \begin{cases} \frac{1}{2}(A-L), & L \leqslant A, \\ 0, & L \geqslant A, \end{cases}$$

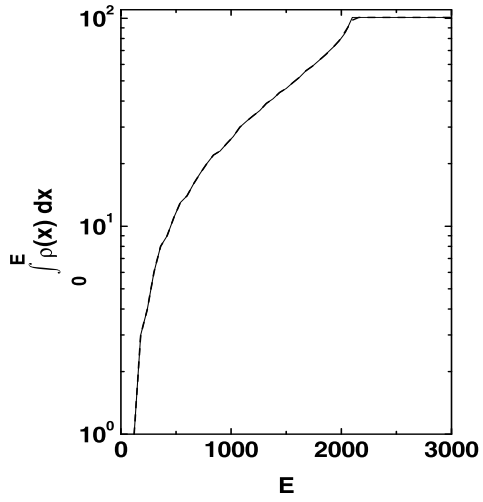

Fig. 1. Integrated density of states, as a function of the energy, corresponding to the Hamiltonians of Eq. (1) (solid line) and of Eq. (12) (dashed line). Both results coincide in the curve shown in the figure. The calculations were performed for $N = 100$, $\omega = 1$, $\lambda = 0.1$, and $\eta = 0.05$. The value $L = 150$ was used. The spectrum of the effective $su_q(2)$ Hamiltonian of Eq. (12) was calculated with $z = 0.08$.

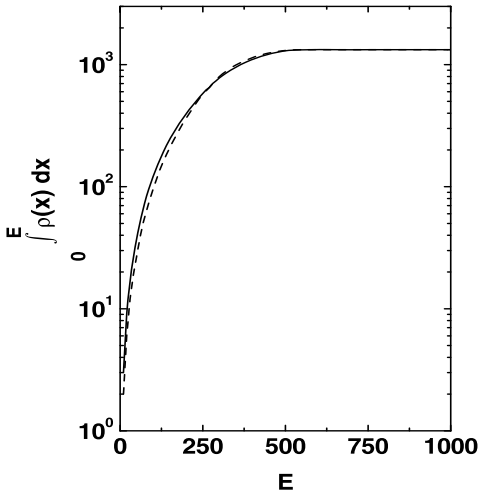

Fig. 2. Integrated density of states, as a function of the energy, corresponding to the Hamiltonians of Eq. (14) (solid line) and of Eq. (27) (dashed line). Both results coincide in the curve shown in the figure. The calculations were performed for $N = 50$, $\omega = 1$, $g_1 = 0.1$, and $g_2 = 0.6$. The value $L = 100$ was used. The spectrum of the effective $su_q(3)$ Hamiltonian of Eq. (27) was calculated with $z = 0.025$.

$$n_{2m} = \begin{cases} \frac{1}{6}(A+L-4), & L \leqslant A,\ 0 < A+L-4 \leqslant A, \\ 0, & L \leqslant A,\ A+L-4 \leqslant 0, \\ \frac{1}{3}(2+L+A) - \frac{1}{3}\sqrt{1+L+L^2+N-NL+N^2}, & L > 0. \end{cases} \tag{28}$$

The Hamiltonian can be diagonalized in the basis of states

$$|n_0 n_1 n_2\rangle = N_{n_0 n_1 n_2} \tilde{b}_0^{\dagger n_0} \tilde{b}_1^{\dagger n_1} \tilde{b}_2^{\dagger n_2} |0\rangle, \tag{29}$$

with $n_1 = A - n_0 - n_2$, and

$$\begin{aligned} &\tilde{N}_i |n_i\rangle = n_i |n_i\rangle, \\ &\tilde{b}_i^{\dagger} |n_i\rangle = \sqrt{[n_i+1]_q}\,|n_i\rangle, \\ &\tilde{b}_i |n_i\rangle = \sqrt{[n_i]_q}\,|n_i\rangle. \end{aligned} \tag{30}$$

3. Results and discussion

3.1. Comparison of exact and q-deformed solutions

In this section we shall explore the consequences of our conjecture by solving, numerically, the eigenvalue problem of H and $H_{(L,q)}$, for the systems discussed in the previous section. To start with we shall demonstrate that our conjecture is indeed supported by the results, that is, we shall calculate the exact solutions and the q-deformed ones, for both the cases of two- and three-level atoms. In Figs. 1–4 we show the results corresponding to the integrated density of states (Figs. 1 and 2) and to the energy spectrum (Figs. 3 and 4), obtained by the exact diagonalization, and by the q-deformed replacement of the interaction terms, of the Hamiltonians describing two- and three-level atoms. In doing so we have chosen some rather large values of L, in order to work in large, but still manageable, spaces. Also, we have calculated states with very large eigenvalues. The results shown in the figures are very illustrative of the situation, although the parameters of the Hamiltonians have been chosen arbitrary, and except for the relative values of the coupling they are not representative of a specific physical system. Both the integrated density of states and the spectra are rather similar and they demonstrate in a self-explanatory manner the validity of our conjecture. Naturally, one may think that, perhaps, the conjecture about the equivalence of the fermion–boson $\rightarrow$ q-deformed-purely-fermionic Hamiltonians may show up in the eigenvalues but with non-comparable eigenfunction (as it should be the case if we would work in a sort of variational scheme). Thus, we test it by looking at spin observables. This is done in the next section, for the case of the spin-squeezing factor for two-level atoms, which is a simple but non-trivial test-case.

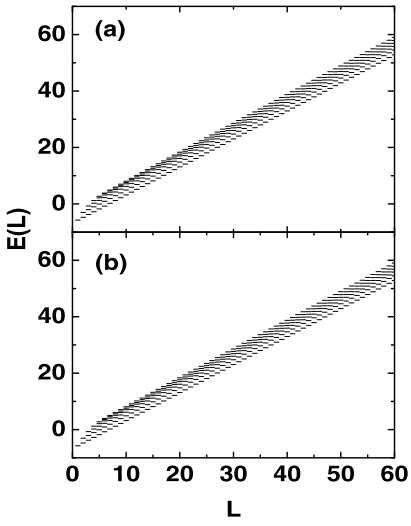

Fig. 3. The spectrum for the $su(2)$ model of Eq. (1) and for the $su_q(2)$ Hamiltonian of Eq. (12). The system consists of $N = 5$ atoms, for $\omega = 1$, $\lambda = 0.1$ and $\eta = 0.05$. The spectrum denoted by (a) corresponds to the one obtained from the Hamiltonian of Eq. (1). The spectrum denoted by (b) is obtained from the effective $su_q(2)$ Hamiltonian of Eq. (12), with $z = 0.29$.

3.2. Application to the calculation of the spin-squeezing factor for two-level atoms

With the corresponding eigenvalues and eigenfunctions, both of the exact and q-deformed cases, we have calculated, by applying Eq. (9), the time-dependent expectation value of the components of the total atomic spin, and their deviations, as required by the definition of the spin-squeezing factor [10]

$$\zeta^2 = 2\frac{(\Delta S_n)^2}{|\langle \boldsymbol{S} \rangle|}, \tag{31}$$

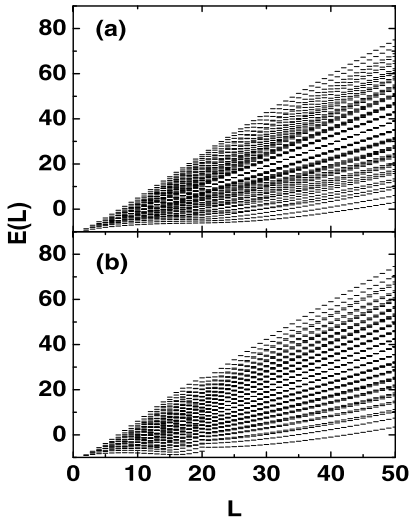

Fig. 4. The spectrum for the $su(3)$ model of Eq. (14) and for the $su_q(3)$ Hamiltonian of Eq. (27), for $N = 10$, $\omega = 1$, $g_1 = 0.1$, and $g_2 = 0.6$. The spectrum denoted by (a) corresponds to the one obtained from the Hamiltonian of Eq. (14). The spectrum denoted by (b) is obtained from the effective $su_q(3)$ Hamiltonian of Eq. (27), with $z = 0.125$.

where N is the number of atoms in the system, and S_n is the component of the total spin in a direction perpendicular to the expectation value $\langle \mathbf{S} \rangle$. Therefore, S_n is oriented in the direction defined by the unitary vector

$$\breve{n} = (\sin\theta_n \cos\phi_n, \sin\theta_n \sin\phi_n, \cos\theta_n), \tag{32}$$

such that $\breve{n} \cdot \langle \mathbf{S} \rangle = 0$.

In the present calculations we have considered the coherent state $|I\rangle$ as the initial condition appearing in Eq. (9). This state is not an eigenstate of H, and it is defined by [10]

$$|I\rangle = \mathcal{N} \sum_{L,k} \alpha_f^k \frac{\alpha_b^{(L-k)}}{\sqrt{(L-k)!}} \binom{N}{k}^{1/2} |L, k\rangle, \tag{33}$$

where $\mathcal{N}$ is a normalization factor; $\alpha_b = \sqrt{n_b}$, being n_b the (externally fixed) mean value of the number of photons in the state, is the factor which characterizes the boson sector of the coherent state, and α_f is the factor corresponding to the atomic sector of the coherent state. The explicit expressions of these factors are the following:

$$\mathcal{N} = \left(1 + |\alpha_f|^2\right)^{-N/2} e^{-\alpha_b/2},$$
$$\alpha_f = e^{-i(\phi_0 - \pi)} \tan(\theta_0/2), \tag{34}$$

ϕ_0 and θ_0 are the orientation angles of the atomic sector of the coherent state.

3.2.1. Exact results

The Hamiltonian of Eq. (1) was diagonalized in the basis (8), making use of the block-symmetry, Eq. (5), with $L \leqslant 80$. The initial state $|I\rangle$ was constructed (see Eq. (33)) with $n_b = 20$ (average number of photons), and with the orientation angles $\theta_0 = \phi_0 = \frac{\pi}{4}$. The value of the coupling constants λ and η, and the energies ω_f and ω_b were fixed at 0.1, 0.05, 1.0, and 1.0, respectively (in arbitrary energy units). These values are arbitrary values, as said before. With the exact eigenvalues and eigenfunctions we have calculated the time-dependent expectation values appearing in the definition of the squeezing factor ζ^2. The results are shown in Fig. 5 (inset (a)) and Fig. 6 (insets (a), (c) and (e)).

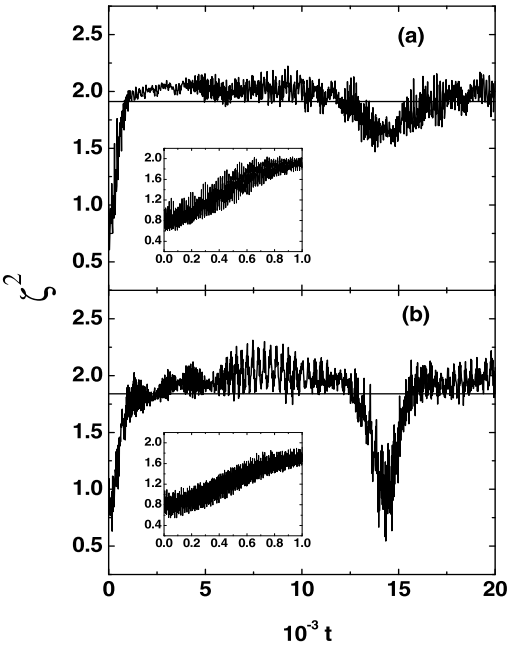

Fig. 5. Time dependence of the squeezing parameter ζ^2, for a system of $N = 5$ two-level atoms. The results have been obtained with the couplings $\lambda = 0.1$ and $\eta = 0.05$ (exact solutions, (a)). The initial state consists of an atomic coherent state (33) with $\theta_0 = \phi_0 = \pi/4$, and of a coherent photon state with an $\langle n_a \rangle = 20$. The time is given in arbitrary units. (b) shows the results corresponding to the effective Hamiltonian of Eq. (12), with a deformation parameter $z = 0.29$. In both figures, the horizontal solid line ($\zeta^2 \approx 1.8$) is the mean value of the squeezing.

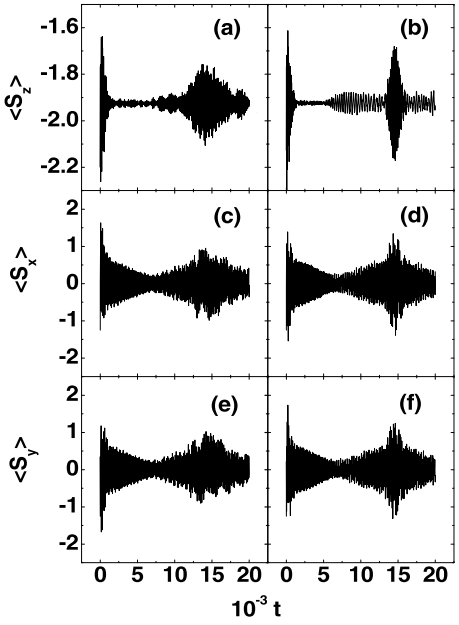

Fig. 6. Mean value of the components of the total atomic spin as a function of time, for the exact solution of H, Eq. (1) (insets (a), (c), (e)), and for the effective Hamiltonian, Eq. (12) (insets (b), (d) and (f)). The calculations have been performed using the parameters given in the captions to Fig. 5.

3.2.2. Results of the q-deformed Hamiltonian

As explained in the text, we have gauged the function $\chi(L, q)$, so that the maximum of the matrix elements of $H_{(L,q)}$ and H become comparable for a fixed value of q [19], and within the same subspace L. In this procedure the value of q is the same for all subspaces. For the present calculations the conditions are fulfilled for $q = 1.336$. With this value of q the diagonalization of $H_{(L,q)}$ yields

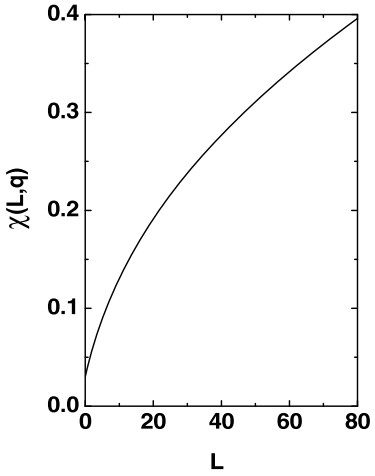

Fig. 7. Dependence of the function $\chi(L,q)$ with L, for the value $q = 1.336$ used in the present calculations.

a spectrum comparable to the one of H. The results, for ζ^2, obtained with $H_{(L,q)}$, are shown in Fig. 5 (inset (b)) and Fig. 6 (insets (b), (d) and (f)).

The similarities between both sets of results are notorious. Not only the values of the squeezing factor ζ^2 agree reasonable for both H and $H_{(L,q)}$, but also the structure of the time dependence of it shows similar features.

In constructing $H_{(L,q)}$, by fixing the scale factor $\chi(L,q)$, we have applied a very crude approach, that is to adjust it to the larger values of the matrix elements of the complete Hamiltonian H, without performing further adjustments. In spite of this, the agreement between both sets of results seemingly confirm the replacement advocated by our conjecture. This is particularly significant for the case of the present calculations, where not only mean values of the spin components are needed but also their fluctuations.

Finally, we have verified that the same similarities appear for different sets of couplings and for different number of atoms. Concerning the function $\chi(L,q)$, it is a smooth function of L, as it is shown in Fig. 7.

4. Conclusions

We have presented two non-trivial cases supporting a conjecture based on the use of quantum algebras, of the class of Hoft algebra, to absorb boson degrees of freedom in an effective, q-deformed, scheme where the fermion–boson interactions are replaced by q-deformed fermion operators. To support this procedure we have compared the density of states and the eigenvalues of the Hamiltonian of two- and three-level atoms interacting with photons. We have shown that the calculation of spin observables, like the squeezing factor, may be greatly simplified by the type of replacement resulting from the application of the conjecture. Work is in progress concerning further applications of the method, as well as on the formal aspects of the conjecture.

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