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Study of squeezing in spin clusters

M. Reboiro^{a,b}, O. Civitarese^{b,*}, L. Rebon^b

^a Faculty of Engineering, University of Lomas de Zamora, Camino de Cintura Km 2, (1836) Lavallol, Argentina
 ^b Department of Physics, University of La Plata, c.c. 67 1900, La Plata, Argentina

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Abstract

The conditions under which spin squeezing occurs in an asymmetric chain of spins are discussed. The time evolution of the system is calculated for different initial conditions. The effects of the use of spin coherent states to model the initial condition are analyzed. © 2007 Elsevier B.V. All rights reserved.

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1. Introduction

The concept of squeezing has been extensively applied to study boson systems [1–3]. In their work, Walls and Zoller [4] have considered the uncertainties between perpendicular spin components. If appropriate quantum mechanical correlations are established among the elementary spins of a system it would be possible to cancel out, partially, the spin fluctuations in one direction at the expense of enhancement in other directions. This could be achieved while preserving the minimum uncertainty between components.

In the last years considerable effort has been devoted to the study of spin squeezing in nonlinear systems. For a general review see [5]. In the following, and for the sake for completeness, we shall quote some of the topics discussed in connection with spin squeezing. Ref. [6] addresses the generation of spin squeezing in cold atoms linked to the improvement of cesium clock performance. In Refs. [7,8] it is shown that the preparation of correlated input states (spin squeezed states) enlarges the signal-to-noise ratio in experiments which are limited by quantum-mechanical uncertainty in the measurement of the

Corresponding author. *E-mail address:* civitare@fisica.unlp.edu.ar (O. Civitarese).

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level populations. In Refs. [9–11] the relation of entanglement and spin squeezing has been analyzed, both from the theoretical and experimental points of view. The study of spin orientation in nonlineal spin-coherent states is found in [12,13]. Optimally squeezed spin states have been considered in [14]. Discussion of a generalized definition of spin squeezing can be found in [15]. Spin squeezing in anisotropic Ising-like models [16,17] has received attention, too, in connection with quantum computing. Recently, spin clusters have been proposed as candidate system for qubits. In Ref. [18,19] it has been shown that a spin chain of an odd number of s = 1/2 spins interacting via antiferromagnetic couplings is a realization of a logical qubit. The spectrum of these systems is characterized by an $S = \frac{1}{2}$ ground state doublet which exhibits an energy gap. The $S = \frac{1}{2}$ ground state doublet can be defined as the cluster qubit. A novel class of molecular magnetic systems has been presented in [20-22]. The quantum hardware is thought as a collection of coupled molecules, each corresponding to a different qubit. The advantage of this scheme, as compared to others based on single-spins encoded as qubits, is the dimensionality of the physical subsystem. It also benefits from the reduction of the spatial resolution required for the selective addressing of each qubit, which is achieved by means of local magnetic fields.

In this work, we discuss the occurrence of spin squeezing in a chain of interacting spins. The calculations are based on the formalism advanced in our previous publication [23]. We analyze the dependence of spin squeezing upon the coupling constants of the interaction, for different number of atoms, and upon the preparation of the initial state. The details of the formalism are presented in Section 2. In Section 3 we present and discuss the solutions for different parameters of the model and different initial conditions. Conclusions are drawn in Section 4.

2. Formalism

2.1. The Hamiltonian

We shall study a cluster formed by an asymmetric spin s = 1/2 chain, described by the Hamiltonian

$$H = \sum_{l,\gamma} g_{\gamma,l} s_{\gamma,l} s_{\gamma,l+1}.$$
 (1)

The subindex l ($l = 1 \rightarrow N_{\text{max}}$) labels the spin site, $\gamma = x, y, z$ reads for spatial directions, $g_{\gamma,l}$ is the coupling constant for the interaction between the spins located at the sites l and l + 1 in the direction γ . N is the number of spins, $N_{\text{max}} = N$ for a close chain or $N_{\text{max}} = N - 1$ for an open chain.

The Hamiltonian of Eq. (1) is diagonalized in the basis

$$|k\rangle = |k_1, k_2, \dots, k_l, \dots, k_N\rangle = \prod_{l=1, k_l=0, 1}^N s_{+, l}^{k_l} |\rangle.$$
 (2)

In the above state, the vacuum $|\rangle$ is the state with all spins with $s_z = -1/2$, and the operators $s_{\pm,l} = s_{x,l} \pm i s_{y,l}$ are the spin ladder operators of the site *l*. Together with $s_{z,l}$ they obey su(2) commutation relations.

The matrix $H_{k,k'}$ is a band matrix, whose matrix elements are given by the expressions

$$\langle k|H|k' \rangle = \delta(k,k') \sum_{l} g_{z,l}(k_{l} - 1/2)(k_{l+1} - 1/2) + \frac{1}{2} \sum_{l} g_{+,l} \delta(k'_{l}, k_{l} + 1) \delta(k'_{l+1}, k_{l+1} - 1) + \frac{1}{2} \sum_{l} g_{+,l} \delta(k'_{l}, k_{l} - 1) \delta(k'_{l+1}, k_{l+1} + 1) + \frac{1}{2} \sum_{l} g_{-,l} \delta(k'_{l}, k_{l} + 1) \delta(k'_{l+1}, k_{l+1} + 1) + \frac{1}{2} \sum_{l} g_{-,l} \delta(k'_{l}, k_{l} - 1) \delta(k'_{l+1}, k_{l+1} - 1),$$
(3)

with $g_{\pm,l} = \frac{1}{2}(g_{x,l} + \pm g_{y,l})$. The eigenvectors *H* are written as linear combinations of the states of the basis {*k*} of Eq. (2)

$$|\alpha\rangle = \sum_{\{k\}} c_{\alpha,k} |k\rangle.$$
(4)

2.2. The time evolution

The time evolution of a given operator O is expressed as

$$O(t) = U^{\dagger}(t)OU(t), \quad U(t) = e^{-iHt/\hbar}.$$
 (5)

The expectation value $\langle O(t) \rangle$ is then written

$$\langle O(t) \rangle = \operatorname{Tr}(\rho(t)O)$$

= $\sum_{\alpha,\beta} \langle \beta | I \rangle \langle I | \alpha \rangle \langle \alpha | O | \beta \rangle e^{-i(E_{\alpha} - E_{\beta})t/\hbar},$ (6)

where the density matrix $\rho(t)$ has been defined as

$$\rho(t) = U^{\dagger}(t)\rho U(t), \quad \rho = |I\rangle\langle I|.$$
(7)

In the above equations $|I\rangle$ is the initial state of the system, $\{E_{\alpha}\}$ and $\{|\alpha\rangle\}$ are the α th eigenvalue and eigenvector of the Hamiltonian.

The expression (6) can be written in a more compact form in terms of the overlap of the initial state $|I\rangle$ with the eigenvectors $\{|\alpha\rangle\}$,

$$\langle O(t) \rangle = \sum_{n,m} T^*(n) \langle n | O | m \rangle T(m),$$

$$T(m) = \sum_{\alpha n} c^*_{\alpha n} c_{\alpha m} \langle n | I \rangle e^{i E_{\alpha} t/\hbar}.$$
(8)

For details see Ref. [23].

2.3. The initial condition

We shall study the time evolution of the total spin operator under the action of H and by adopting a coherent spin state (CSS) as the initial condition $|I\rangle$ [13,24]. A CSS is defined as an eigenstate of the spin component, **S** . **n**₀, in the direction of a unit vector **n**₀, that is

$$\mathbf{S} \cdot \mathbf{n}_0 |\mathbf{CSS}\rangle = -S |\mathbf{CSS}\rangle \tag{9}$$

where $S = \sum_{i} s_{z}(i)$. In the following, we shall denote by *S* the eigenvalue of the spin in the direction of **n**₀.

The orientation of \mathbf{n}_0 on the sphere is given by the polar and azimuthal angles θ_0 and ϕ_0 . Thus, $\mathbf{n}_0(\theta_0, \phi_0) = (\sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0, \cos \theta_0)$.

The initial condition is written as

$$|I\rangle = e^{zS_{+}}|\rangle,$$

= $(1+|z|^{2})^{-\frac{N}{2}} \sum_{m=0}^{N} \frac{z^{m}}{m!} S_{+}^{m}|\rangle,$ (10)

with

$$z = e^{-i(\phi_0 - \pi)} \tan\left(\frac{\theta_0}{2}\right). \tag{11}$$

The operator S_+ is the total spin raising operator

$$S_{+} = \sum_{l} s_{+,l}.$$
 (12)

Atomic spin-squeezed states are quantum correlated systems with reduced fluctuations in one of the components of the total spin. Let us consider the component S_n of the total spin S along the unit vector **n** such that $\mathbf{n} \cdot \langle \mathbf{S} \rangle = 0$, and let \mathbf{S}_{\perp} be the component of the spin in the direction perpendicular to both **n** and $\langle \mathbf{S} \rangle$. The quantity

$$\zeta^2 = \frac{2(\Delta S_n)^2}{|\langle \mathbf{S} \rangle|} \tag{13}$$

measures the ratio of the spin fluctuations in the **n**-direction and the mean value of the spin. If one applies commutation relations among the components of the spin one also gets $\zeta^2 = \frac{\Delta S_n}{\Delta S_{\perp}}$. Following Refs. [5,13] we shall say that **S**_n is squeezed if $\zeta^2 < 1$. Physically it means that the mean value of the spin **S** precesses in the plane determined by the orientations **S**_n and **S**_{\perp}. The precession goes along a curve with semi-axes ΔS_n and ΔS_{\perp} in a plane perpendicular to the direction of $\langle S \rangle$.

3. Results and discussion

We have diagonalized the Hamiltonian of Eq. (1), and studied the time evolution of the system for different initial states, that is CSS states prepared at different orientations (θ_0 , ϕ_0). The aim of the calculations is to investigate the time evolution of the unit vector **n**, which is the unit vector perpendicular to the instantaneous direction of the mean value of the spin $\langle \mathbf{S} \rangle$, and the time evolution of the unit vector $\mathbf{n}_{\mathbf{S}} = \frac{\langle \mathbf{S} \rangle}{|\langle \mathbf{S} \rangle|}$. With these elements, we may calculate the localization of the squeezing factor ζ^2 in the plane determined by the orientation angles (θ , ϕ) of **n**. We have taken the number of spin sites and the coupling constants $g_{z,l}$ and $g_{\pm,l}$ as free parameters, both for open and close chains.

The results of our calculations are shown in Figs. 1–8, which illustrate the effects of: (a) the orientation of the CSS, (b) the asymmetry of the couplings between the spin sites, (c) the number of spin-sites, (d) the localization of the squeezing factor ζ^2 . These are results which may eventually be used in further comparison with experimental results, but for the sake of the present work we have taken them as indicatives of the sensitivity of ζ^2 upon the prepared initial conditions and upon the details of the spin chain. The actual values of the couplings are only indicative values with no physical justification, and they have been chosen in a way such that the asymmetry of the couplings becomes notorious. The following is a brief description of the main features exhibited by the results.

Fig. 1 shows the time evolution of the direction of the unit vector \mathbf{n}_S . The effect of the asymmetry of the adopted couplings (see the caption to Fig. 1) reflects upon the fast variations of the precession around the *z* axis. That is the fast variation of the azimuthal angle and the narrowing of the orbit given by nearly constant value of θ . For these cases, the CSS is pointing out at the ($\pi/4$, $\pi/4$) direction while the spin rotates around the *z* axis pointing at the direction given by $\theta = 3\pi/4$. Physically speaking it means that the measured direction of the total spin may be related to the actual couplings of the spin chain, for a fixed initial condition.

Fig. 2 shows the time evolution of the squeezing factor ζ^2 in the (θ, ϕ) -plane. The system is the same of Fig. 1 (open chain), with the parameters of case Fig. 1(a). The results display a degree of periodicity in the occurrence of squeezing. It is clearly

Fig. 1. Time evolution of the direction of the unit vector $\mathbf{n}_S = \langle \mathbf{S} \rangle / |\langle \mathbf{S} \rangle|$. The time dependence of the angles θ (solid line) and ϕ (dashed line) is depicted for $0 \leq t \leq 25$, in arbitrary units of time. The system is an open chain of N = 5 sites. The initial state is a CSS defined by $\theta_0 = \phi_0 = \pi/4$. The coupling constants are $g_{z,l} = 4$, $g_{+,l} = 1$, $g_{-,l} = 0$, for all *l* values (case (a)) and $g_{z,l} = 4$ if *l* is odd, 8 for l = 2 and 16 for l = 4, $g_{+,l} = 1$ (*l* odd), 2 (*l* = 2), 4 (*l* = 4) and $g_{-,l} = 0$ (case (b)). In this figure, and in the subsequent ones, the CSS has an overall phase $-e^{i\phi_0}$.

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seen, from the results shown in the Fig. 2, that the regions where $\zeta^2 < 1$ appear and disappear at approximately $\Delta t = 10$. This is the interval of time for which the maximal squeezing is obtained.

The effect of the asymmetry of the interaction upon squeezing is shown in Fig. 3. The comparison with the results of Fig. 2 suggests the suppression of squeezing due to the asymmetry of the interaction. This may be naively understood by taking the





Fig. 3. The same as Fig. 2 for the parameters corresponding to case (b) of Fig. 1.



Fig. 4. Dependence of the squeezing factor ζ^2 upon the initial CSS. In this figure all cases correspond to t = 4, for an open chain with N = 5 and $g_{z,l} = 4$, $g_{+,l} = 1$, $g_{-,l} = 0$, for all *l* values. The parametrization of the CSS (θ_0 , ϕ_0) is the following: (a) ($\pi/2$, 0), (b) ($\pi/4$, 0), (c) (0, 0), (d) ($\pi/8$, 0), (e) ($\pi/8$, $\pi/4$), (f) ($\pi/8$, $\pi/2$). The meaning of the contour plots is explained in the caption of Fig. 2.

asymmetry as a small perturbation δ in the *z*-direction. It is then seen that the amplitude of the wave function is reduced by a factor $(1. - \delta)$, which suppresses the mean value of the total spin.

The dependence of ζ^2 upon the parameters of the initial condition (CSS) is very strong. The set of results shown in Fig. 4 seemingly indicates that $\zeta^2 < 1$ for CSS with $\theta_0 \neq 0$. For the CSS with $\theta_0 = \phi_0 = 0$, the vector **n** is oriented by $\theta = \pi/2$ for all values of ϕ , and this is the limiting case for which $\zeta^2 = 1$ in all the plane. The effect of the number of sites upon ζ^2 is show in Fig. 5 (symmetric interactions) and Fig. 6 (non-symmetric interactions). In both figures the CSS has been defined with



Fig. 5. Effect of change in the number of spin sites, for an open chain, upon ζ^2 . The time is t = 4, the parameters of the CSS are $\theta_0 = \phi_0 = \pi/4$ and the couplings are $g_{z,l} = 4$, $g_{+,l} = 1$, $g_{-,l} = 0$, for all *l* values. Insets (a), (b) and (c) show the results for N = 5, 6, 7 sites, respectively.



Fig. 6. Effect of change in the number of spin sites, for an open chain, upon ζ^2 . The time is t = 4, the parameters of the CSS are $\theta_0 = \phi_0 = \pi/4$ and the couplings are $g_{z,l} = 4$ (*l* odd), 16 (*l* even), $g_{+,l} = 1$ (*l* odd), 4 (*l* even), $g_{-,l} = 0$ for all *l* values. Insets (a), (b) and (c) show the results for N = 5, 6, 7 sites, respectively.

 $\theta_0 = \phi_0 = \pi/4$ and t = 4. It is seen that, for this relatively small time scale, the asymmetry of the interaction influences more the behavior of ζ^2 than the change of the number of spins in the chain.

So far we have considered open spin chains. The comparison of results for open and close spin chains is shown in Figs. 7 and 8. The initial CSS state is the same for both figures, as well as the couplings. The number of spin sites was taken as variable, 5 sites for the case of Fig. 5, and 7 sites for the case of Fig. 7. Although the variation with time of ζ^2 is larger for the case of a close spin chain, as compared with the more narrow spreading displayed by the results of the open spin chain, close chain seems to exhibit a maximal squeezing. The time scale of these two last figures is larger than the one of the previous figures, but the tendency shown by the results of Figs. 1–6 persists. The comparison of the results of Figs. 7 and 8 also shows that squeezing is washed-out if the number of sites is increased.

The general feature which emerges from these results is the possibility to constraint the motion and the fluctuations of the total spin, on the sphere, by

- (i) changing adequately the parameters of the interactions between spin sites,
- (ii) preparing the initial condition, and
- (iii) activating open or close chains with variable number of spin sites.



Fig. 7. Results for N = 5 atoms. Insets (a) and (c) correspond to the open chain case, while insets (b) and (d) correspond to the close chain case. The couplings are: $g_{z,l} = 4$ for (a) and (b), and $g_{z,l} = 4$ (if *l* is odd) and $g_{z,l} = 16$ (if *l* is even) for cases (c) and (d); $g_{+,l} = 1$ for (a) and (b), and $g_{+,l} = 1$ (if *l* is odd) and $g_{+,l} = 4$ (if *l* is even) for (c) and (d), $g_{-,l} = 0$ for all cases. The insets show the maximal value of the squeezing as a function of time.



Fig. 8. Results for N = 7 atoms. The couplings are given in the captions to Fig. 7.

4. Conclusions

In this work we have studied the occurrence of squeezing in spin clusters. We have considered open spin chains and taken the fluctuations of the total spin in space directions as an indicator of squeezing. The calculations have been performed for different sets of parameters of the Hamiltonian and of the initial CSS state. We summarize the results of our calculations in the following: (i) spin squeezing in an open chain is reduced if the interactions between the spin sites are non-symmetric (anisotropic), (ii) the dependence of spin squeezing upon the initial conditions, in our case a CSS state, is indeed very strong. These conclusions are limited, of course, by the model parameters introduced in the text. They are, nonetheless, illustrative of the trend which may appear in a more involved case. Work is in progress concerning the description of squeezing in fermion– boson system with spin–spin interactions included [25].

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References

- [1] D.F. Walls, Nature 306 (1983) 141;
- R. Loudon, P.L. Knight, J. Mod. Opt. 34 (1987) 709; V.V. Dodonov, J. Opt. B: Quantum Semiclass. Opt. 4 (2002) R1.
- M.O. Scully, M.S. Zubairy, Quantum Optics, Cambridge Univ. Press, Cambridge, 1997;
 J.S. Peng, G.X. Lee, Introduction to Modern Quantum Optics, World Sci-
- entific, Singapore, 1999.
- [3] B.R. Mollow, R.J. Glauber, Phys. Rev. 160 (1967) 1076.
- [4] D.F. Walls, P. Zoller, Phys. Rev. Lett. 47 (1981) 709.
- [5] P.D. Drummond, Z. Ficek, Quantum Squeezing, Springer, Berlin, 2004.
- [6] D. Oblack, et al., Phys. Rev. B 71 (2005) 043807.
- [7] D.J. Wineland, J.J. Bollinger, W.M. Itano, F.L. Moore, D.J. Heinzen, Phys. Rev. A 46 (1992) R6797.
- [8] D.J. Wineland, J.J. Bollinger, W.M. Itano, D.J. Heinzen, Phys. Rev. A 50 (1994) 67.
- [9] A. Messikh, Z. Ficek, M.R.B. Wahiddin, J. Opt. B: Quantum Semiclass. Opt. 5 (2003) L1.
- [10] J. Hald, J.L. Sørensen, C. Schori, E.S. Polzik, Phys. Rev. Lett. 83 (1999) 1319.
- [11] V. Josse, A. Dantan, A. Bramatti, E. Giacobino, J. Opt. B: Quantum Semiclass. Opt. 6 (2004) S532.
- [12] X. Wang, J. Opt. B: Quantum Semiclass. Opt. 3 (2001) 93.
- [13] M. Kitagawa, M. Ueda, Phys. Rev. A 47 (1993) 5138.
- [14] A.G. Rojo, Phys. Rev. A 68 (2003) 013807.
- [15] H. Prakash, R. Kumar, J. Opt. B: Quantum Semiclass. Opt. 7 (2005) S757.
- [16] A. Sørensen, K. Møler, Phys. Rev. Lett. 83 (1999) 2274.
- [17] X. Wang, A. Sørensen, K. Møler, Phys. Rev. A 64 (2001) 053815.
- [18] F. Meier, J. Levy, D. Loss, Phys. Rev. Lett. 90 (2003) 047901.
- [19] F. Meier, J. Levy, D. Loss, Phys. Rev. B 68 (2003) 134417.
- [20] F. Troinai, et al., Phys. Rev. Lett. 94 (2005) 207208.
- [21] M. Affronte, et al., Appl. Phys. Lett. 84 (2004) 3468.
- [22] S. Carretta, et al., Phys. Rev. B 67 (2003) 094405.
- [23] O. Civitarese, M. Reboiro, Phys. Lett. A 357 (2006) 224.
- [24] R.J. Glauber, Phys. Rev. 130 (1963) 2529;
 - R.J. Glauber, Phys. Rev. 131 (1963) 2766.
- [25] O. Civitarese, M. Reboiro, L. Rebon, in preparation.