The Friedrichs-Model with fermion-boson couplings

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Abstract: In this work we present an extended version of the Friedrichs Model, which includes fermion-boson couplings. The set of fermion bound states is coupled to a boson field with discrete and continuous components. As a result of the coupling some of the fermion states may become resonant states. This feature suggests the existence of a formal link between the occurrence of Gamow Resonant States in the boson sector, as predicted by the standard Friedrichs Model, with similar effects in the set of solutions of the fermion central potential (Gamow fermion resonances). The structure of the solutions of the model is discussed by using different approximations to the model space. Realistic couplings constants are used to calculate fermion resonances in a heavy mass nucleus.

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1 Introduction

The conventional approach to the quantum mechanical description of the nuclear many body problem is, of course, the shell model [1]. The task of constructing many-body nuclear wave functions, starting from the shell

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model treatment of two-body interactions, is formidable because of the extremely large dimension of the basis, and only very recently some impressive steps towards that goal have been achieved [2]. Alternatively, one may think of physical oriented approximations to the problem, based on the use of effective one-particle and collective degrees of freedom and their couplings [3, 4]. Central to this picture is the role play by one particle fermion bound states, collective boson excitations (either vibrational or rotational) and particle-boson couplings [4].

In both approaches, either the shell model or the one-particle plus collective excitations, single-particle states are obtained as solutions of nuclear central potentials, like the nuclear harmonic oscillator or the Woods-Saxon potential [3]. From the mathematical point of view these are solutions of the Schroedinger equation with a central potential with volume, surface and spin-orbit interactions [5]. Since the nucleus is a system with a finite density of one-particle states, one is interested in one-particle bound state solutions. Shell model basis are thus composed by these single-particle states [6]. Since the one-particle potentials are of finite range, single-particle resonant states are obtained by imposing purely outgoing boundary conditions [7].

Tore Bergreen [8, 11] was one of the first physicist to show that a single-particle basis may contain not only bound states but also some single-particle resonances. Lately, the work of many others nuclear structure physicist followed [9, 12, 13] and by now the subject is rather well established and several nuclear structure calculations have been performed in Bergreen's basis to describe nuclear structure properties in the continuum, decay widths of particle states and vibrational states, nuclear clusters and their decays [19].

Since the calculations involved a rather large number of states, even in the simplest versions, one may be tempted to study the effects of couplings between bound and resonant states in the context of solvable models, in order to extract signatures of possible dominant mechanisms.

In the last 40 years, the use of solvable models has contributed significantly to the understanding of the nuclear dynamics, in spite of its complexity. Although solvable models are popular in the context of bound single-particle and collective states [16, 14], their use to describe collective and single-particle properties in the continuum is less frequently found in the literature.

The Friedrichs model [20] is a suitable mathematical model to describe the coupling with the continuum. The model has the advantage of being exactly solvable and it can be applied to various systems of physical interest

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[21, 22, 23, 24]. In its simplest version the model deals with the coupling between discrete and continuous components of a boson field [20]. More sophisticated versions of the Friedrichs model have been presented in [21, 22, 26]. It is the only completely solvable model which describes resonance phenomena [27]. A recent presentation of the model, in connection with the study of resonances in field theory, can be found in [23]. In this work we aim at the extension of the model, in order to include the coupling between fermion and boson fields.

The advantage of using an exactly solvable model, like the Friedrichs model, to study the coupling with the continuum is rather obvious, since until now the description of these couplings are limited to purely numerical approaches. In the standard approach, like the one advocated in the literature [9, 14, 13], the occurrence of resonances in the fermion sector is a result of the boundary conditions imposed to the diagonalization of finite range central potentials. Purely outgoing boundary conditions lead to Gamow resonances; some of these states, depending upon the imaginary part of their energies, may be included in single-particle basis, the so-called Berggren basis, together with fermion bound-states. In this hibrid basis one may attempt to include residual two-body interactions. As shown [13] in the energies of some particle-hole excitations may become complex. The same effect shows up at the level of attractive two-particle configurations [28, 17], where resonances may appear as a consequence of the coupling between configurations of fermion pairs in bound and resonant states. Finally, it is worth mentioning that the observed properties of atomic nuclei in the drip line may be strongly linked to the coupling between bound and resonant states. Particularly, we refer to the broadening of the single particle energies and to the appearance of a decay width in single-particle orbits, which may be strongly influenced by the coupling to continuous particle-hole excitations [17, 18]. A possible mechanism to explain for this class of phenomena is the coupling between bound fermion states and discrete and continuous bosonic degrees of freedom.

Since the standard Friedrichs model describes the coupling between the discrete and the continuous components of a single boson field, it seems natural to include fermions and their couplings with the boson field and take advantage of the solvable structure of the model. In this respect, the results of the extended Friedrichs model may be used to approximate some specific features of fermion states, like their decay width, in situations where the coupling between fermions and bosons indicates the existence of effects due to the continuum [18].

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The present paper is organized as follows: In Section 2, we present the needed mathematical details about the model. In Section 3, we explore the structure of the solution obtained under different approximations. We have applied a compact formalism to solve the system of coupled equations yielded by the model. The results of applications, with reference to a realistic case, are presented and discussed in Section 3. Conclusions are drawn in Section 4.

2   The Model

For the benefit of the reader, we shall begin with a review of the basic ideas of the Friedrichs Model, which is an exactly solvable model where a discrete boson state interacts with a continuum spectrum of boson excitations. The situation resembles very much that of the nuclear excitations of even mass nuclei near the drip line, where few bound states may strongly interact with low-lying continuum states. This model provides the insight about the mechanism leading to the appearance of resonances in the boson sector, and it has the clear benefit of being solvable.

Next, we shall review briefly the properties of resonant solutions of the Friedrichs model and then we shall proceed with the description of the extension which we propose in this paper. It consists of the inclusion of fermion bound states which interact with the boson resonant state. Since the extended model is also solvable, one might also get some insight about the effects of the coupling of boson and fermion states in the vicinity of the nucleon drip line. From the mathematical point of view the extension of the Friedrichs model amounts to the simultaneous treatment of fermion and boson sectors of a Hamiltonian which includes resonant states in its spectrum. From the point of view of physics it represents the solvable limit of nucleon-boson interactions in Berggreen's basis [18].

In spite of some formal steps of the derivation, the present discussion is motivated by the need to build a schematic, albeit solvable, model which may be use to approach dominant features of the particle-vibration coupling in the continuum.

Further details about the formalism concerning the standard Friedrichs model can be found in [29, 31] and references quoted therein.

2.1 The Standard Friedrichs Model

The simplest form of the Friedrichs model [20] includes a free Hamiltonian H_0 with a real positive continuous spectrum ($\omega > 0$) and a discrete eigenvalue ($\omega_0 > 0$) imbedded in the continuous spectrum. An interaction is acting between the continuous and discrete parts of H_0 by means of a potential V. As a result of the action of V, the bound state of H_0 is dissolved in the continuous and a resonance is produced. The spectrum of the total Hamiltonian $H = H_0 + \lambda V$, where λ is a real coupling constant, is purely continuous and coincides with the real positive semiaxis. We write for the free Hamiltonian, H_0, the expression

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega|\, d\omega\,. \tag{1}$$

The interaction, V, is given by

$$V = \int_0^\infty [V^*(\omega)\, |\omega\rangle\langle 1| + V(\omega)\, |1\rangle\langle\omega|]\, d\omega\,, \tag{2}$$

where, $V(\omega)$ is called the form factor [22, 30].

The most general form of the state vector ψ, in the basis spanned by $\{|1\rangle, |\omega\rangle\}$, can be written as

$$\psi = \alpha|1\rangle + \int_0^\infty \varphi(\omega)\, |\omega\rangle\, d\omega. \tag{3}$$

Note that the set $\{|1\rangle, |\omega\rangle\}$, $\omega \in [0, \infty)$, of generalized eigenvectors of H_0, is a complete set. Thus, each vector state ψ has a component on the bound state $|1\rangle$ of H_0 and an infinite number of components, in the continuous part of the spectrum of H_0. The action of V on ψ is given by

$$V\psi = \left(\int_0^\infty d\omega V(\omega)\varphi(\omega)\right) |1\rangle + \alpha \int_0^\infty d\omega V^*(\omega)|\omega\rangle\,. \tag{4}$$

To obtain the explicit form of (3), that is to determine the amplitudes α and $\varphi(\omega)$, we shall use the technique presented in [31][1]. In the notation of [31], the reduced resolvent, $\eta(z)$ is given by (see also [20, 30])

$$\eta(z) = \langle 1|\frac{1}{H - z}|1\rangle\,. \tag{5}$$

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[1]See also [24], section 3.5.

By making adequate assumptions on the form-factor $V(\omega)$, $\eta(z)$ has the following properties [24]:

i.) The function $\eta(z)$ has no singularities in the complex plane other than a branch cut coinciding with the positive semi-axis, i.e., the continuous spectrum of H.

ii.) The function $\eta(z)$ can be analytically continued through the cut. These are the functions $\eta_\pm(z)$

$$\eta_\pm(z) = \langle 1|\frac{1}{H - \omega \pm i0}|1\rangle = \omega_0 - z - \int_0^\infty \frac{\lambda\,|V(\omega)|^2\,d\omega}{z - \omega \pm i0}\,. \tag{6}$$

These extensions have poles located at the points $z_0 = E_R - i\frac{\Gamma}{2}$, for $\eta_-(z)$, and z_0^*, for $\eta_+(z)$, with $E_R > 0$ and $\Gamma > 0$.

Although resonance poles may, in principle, have arbitrary multiplicity [32], depending on $V(\omega)$, we shall assume hereafter that the resonance poles are simple.

2.2 Resonance or Gamow states.

Poles of (6) appear into complex conjugate pairs, each of them represents a resonance. Then, Gamow vectors are solutions of the eigenvalue equation,

$$(H - x)\,\Psi(x) = 0\,. \tag{7}$$

with the total Hamiltonian H, whose eigenvalues coincide with resonance poles [31].

Since $|1\rangle$ and $|\omega\rangle$ form a complete system, we must have:

$$\Psi(x) = \alpha(x)\,|1\rangle + \int_0^\infty \psi(x,\omega)\,|\omega\rangle\,d\omega\,. \tag{8}$$

If we apply H to (8), we obtain the following system of equations:

$$(\omega_0 - x)\alpha(\omega) + \lambda \int_0^\infty \psi(x,\omega)\,V^*(\omega)\,d\omega = 0\,, \tag{9}$$

$$(\omega - x)\,\psi(x,\omega) + \lambda V(\omega)\,\alpha(\omega) = 0\,. \tag{10}$$

To solve this system, we write $\alpha(\omega)$ in terms of $\psi(x,\omega)$ using (10) and carry the result to (9). We obtain an integral equation which gives one solution of the form:

$$\Psi_+(x) = |x\rangle + \lambda V^*(x) \frac{1}{\eta_+(x)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{V(\omega)}{x - \omega + i0} |\omega\rangle \, d\omega \right\}. \tag{11}$$

The bracket $\Psi_+(x)$, with an arbitrary vector of the form ψ, gives a function which is analytically continuable from above to below through the spectrum of the Hamiltonian H. This continuation has a simple pole at z_0 so that we can write on a neighborhood of z_0:

$$\Psi_+(z) = \frac{C}{z - z_0} + o(z). \tag{12}$$

From (7) and (12), we get

$$(H - z)\Psi_+(z) = \frac{1}{z - z_0}(H - z)C + (H - z)o(z) = 0, \tag{13}$$

which gives

$$(H - z_0)C = 0 \Longrightarrow HC = z_0 C. \tag{14}$$

This shows that the residue C of $\Psi_+(z)$ at the pole z_0 coincides, save for an irrelevant constant, with the decaying Gamow vector $|f_0\rangle^2$. To calculate its explicit form note that (12), on a neighborhood of z_0, has the form:

$$\Psi_+(z) \approx \frac{\text{constant}}{(z - z_0)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{V(\omega)}{z - \omega + i0} |\omega\rangle \, d\omega \right\} + \text{RT}, \tag{15}$$

where RT stand for "regular terms". Now, let us use the Taylor theorem to write:

$$\frac{1}{z - \omega + i0} = \frac{1}{z_0 - \omega + i0} - \frac{z - z_0}{(z_0 - \omega + i0)^2} + o(z). \tag{16}$$

By replacing (16) in (15), we get

$$\Psi_+(z) \approx \frac{\text{constant}}{(z - z_0)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{V(\omega)}{z_0 - \omega + i0} |\omega\rangle \, d\omega \right\} + \text{RT}. \tag{17}$$

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[2]Each resonance is characterized by a pair of complex conjugate poles, $z_0 = E_R - i\Gamma/2$ and $z_0^* = E_R + i\Gamma/2$, in the continuation of $\eta(z)$ from above to below, $\eta_-(z)$, and from below to above, $\eta_+(z)$, respectively. Both are eigenvalues of H and the corresponding eigenvectors are the decaying Gamow vector, $|f_0\rangle$, with z_0, and the growing Gamow vector, $|\widetilde{f}_0\rangle$, with z_0^*.

Therefore, up to an irrelevant constant, we conclude that

$$C \equiv |f_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda V(\omega)}{z_0 - \omega + i0} |\omega\rangle \, d\omega \,. \tag{18}$$

The system given by equations (9) and (10) has another solution that can be analytically continued in the upper half plane. This solution gives, using the same technique, the growing Gamow vector $|\widetilde{f_0}\rangle$:

$$|\widetilde{f_0}\rangle = |1\rangle + \int_0^\infty \frac{\lambda V^*(\omega)}{z_0^* - \omega - i0} |\omega\rangle \, d\omega \,. \tag{19}$$

So far, we have introduced the simplest and more standard version of the Friedrichs model in a fashion accessible to the reader. This will facilitate the comprehension of the more complicated version under consideration in the next subsection.

2.3   The extended Friedrichs Model.

The present version of the Friedrichs model, is an extension of the previously introduced standard Friedrichs model of Section 1.2 and it includes :

i) The unperturbed fermion and boson Hamiltonian H_I

$$H_I = \omega_0 |1\rangle\langle 1| + \int_0^\infty d\omega \omega |\omega\rangle\langle\omega| + \sum_k c_k |k\rangle\langle k| \,, \tag{20}$$

where the index k runs out the set of Fermion kets $|k\rangle$.

ii) The interaction between fermions, $|k\rangle$, and the discrete boson, $|1\rangle$, Hamiltonian H_{II}:

$$H_{II} = \sum_{k,l} \left[h_{k,l} |k,1\rangle\langle l| + h_{k,l}^* |l\rangle\langle k,1| \right] \,. \tag{21}$$

iii) The interaction between fermions and the boson field, $\{|\omega\rangle\}$, H_{III}:

$$H_{III} = \sum_{k,l} \int_0^\infty d\omega \left[f_{k,l}(\omega) |k,\omega\rangle\langle l| + f_{k,l}^*(\omega) |l\rangle\langle k,\omega| \right] \,. \tag{22}$$

The standard Friedrichs model includes the boson-boson coupling V described by equation (2). This coupling V can be generalized to include fermion-boson interactions in the following manner [4, 33]

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$$H_{IV} = \sum_{k,k'} \int_0^\infty d\omega \left[g_{kk'}(\omega)|k,1\rangle\langle k',\omega| + g_{kk'}^*(\omega)|k',\omega\rangle\langle k,1| \right]. \quad (23)$$

To show that H_{IV} generalizes V, of Eq.(2), it is enough to replace the coupling $g_{kk'}(\omega)$ by $g(\omega)$ for all values of k and use the identity $\sum_k |k\rangle\langle k| = 1$.

In the following, we shall use $g(\omega) = V(\omega)$, for which H_{IV} coincides with the standard Friedrichs model interaction V.

To obtain the solution of the eigenvalue problem

$$(H - E)\Psi(E) = 0, \quad (24)$$

where $H = H_I + H_{II} + H_{III} + V$, we write $\Psi(E)$ in its most general form, as

$$\Psi(E) = \sum_k \varphi_k(E)|k\rangle + \sum_k \phi_{k,1}(E)|k,1\rangle + \sum_k \int_0^\infty d\omega\, \psi_k(E,\omega)|k,\omega\rangle. \quad (25)$$

The action of the Hamiltonian on $\Psi(E)$ yields

$$(H - E)\Psi(E) \equiv \sum_k \varphi_k(E)(c_k - E)|l\rangle + \sum_{k,l} h_{kl}^* \phi_{k,1}(E)|k\rangle$$
$$+ \sum_k (c_k + \omega_0 - E)\phi_{k,1}(E)|k,1\rangle + \sum_k \int_0^\infty d\omega\psi_k(E,\omega)V(\omega)|k,1\rangle$$
$$+ \sum_k \phi_{k,1}(E)\int_0^\infty d\omega V^*(\omega)|k,\omega\rangle + \sum_k \int_0^\infty d\omega\psi_k(E,\omega)(c_k + \omega - E)|k,\omega\rangle$$
$$+ \sum_{k,l} \int_0^\infty d\omega\psi_k(E,\omega)f_{kl}^*(\omega)|l\rangle + \sum_{i,k} \varphi_k(E)h_{ik}|i,1\rangle$$
$$+ \sum_{i,k} \int_0^\infty d\omega\varphi_k(E)f_{ik}(\omega)|i,\omega\rangle = 0 \quad (26)$$

The above equation can be re-arranged as a linear combination of $|k\rangle$, $|k,1\rangle$, and $|k,\omega\rangle$ with vanishing coefficients. Thus,

$$\varphi_k(E)(c_k - E) + \sum_l h_{lk}^* \phi_{l1}(E) + \sum_l \int_0^\infty d\omega\psi_l(E,\omega)f_{lk}^*(\omega) = 0 \quad (27)$$

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$$\sum_l \varphi_l(E) h_{kl} + (c_k + \omega_0 - E)\phi_{k1}(E) + \int_0^\infty d\omega \psi_k(E,\omega) V(\omega) = 0 \qquad (28)$$

$$\psi_k(E,\omega)(c_k + \omega - E) + \sum_l \varphi_l(E) f_{kl}(\omega) + \phi_{k1}(E) V^*(\omega) = 0 \qquad (29)$$

for $|k\rangle$, $|k,1\rangle$, and $|k,\omega\rangle$, respectively. A new set of equations, where only the amplitudes $\varphi_l(E)$ and $\phi_{l1}(E)$ appear, can be obtained from the above system. From the last equation we obtain:

$$\psi_k(E,\omega) = c\delta(c_k - \omega - E) - \sum_l \frac{\varphi_l(E) f_{kl}(\omega)}{c_k - \omega - E} - \frac{\phi_{k1}(E) V^*(\omega)}{c_k - \omega - E}, \qquad (30)$$

where c is an arbitrary constant.

If we replace (30) into (27) and (28), we obtain an infinite system of coupled equations in the amplitudes $\phi_{l1}(E)$ and $\varphi_l(E)$. In the most general case, to find the analytic solution of this system may be rather difficult. Note that the resonance behavior is attained by the coupling to the bosons. In order to obtain solutions for the above system, we need to make appropriate choices of the model input represented by the couplings: h_{kl}, $f_{kl}(\omega)$ and $V(\omega)$.

After this substitution, we obtain the following result:

$$\left[(c_k - E)\delta_{km} - \sum_m A_{km}(E) \right] \varphi_m(E)$$
$$+ \sum_m \left(h_{mk}^* - B_{km}(E) \right) \phi_{m1}(E) = -c \sum_m f_{mk}^*(E - c_m) \qquad (31)$$

and

$$\sum_l [h_{kl} - \widetilde{B}_{kl}(E)]\varphi_l(E)$$

$$+(c_k + \omega_0 - E - C_k(E))\phi_{k1}(E) = -cV(E - c_k), \qquad (32)$$

where we have introduced the following notations:

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$$A_{km}(E) = \int_0^\infty d\omega \sum_l \frac{f_{lk}^*(\omega)\, f_{lm}(\omega)}{c_l + \omega - E} \tag{33}$$

$$B_{km}(E) = \int_0^\infty d\omega \frac{f_{mk}^*(\omega)\, V^*(\omega)}{c_m + \omega - E} \tag{34}$$

$$\widetilde{B}_{km}(E) = \int_0^\infty d\omega \frac{V(\omega)\, f_{km}(\omega)}{c_k + \omega - E} \tag{35}$$

$$C_k(E) = \int_0^\infty d\omega \frac{|V(\omega)|^2}{c_k + \omega - E} \tag{36}$$

Notice that expressions (34-36) contain both $f(\omega)$ and $V(\omega)$, as a result of coupling of fermions and bosons in the continuous sector of the model. The solution already found in Sections 2.1 and 2.2 is recovered if $f_{km}(\omega) = h_{km} = 0$, since $C_k(E), \neq 0$ and then Eq.(32) reduces to Eq.(6).

3 A simplified version of the Model: One fermion level.

In this section, we are going to discuss a simplified version of the Friedrichs model including the following ingredients:

i.) A single fermionic bound level. This means that there is only one term in the sum that appears in H_I in (20). This term is denoted as

$$H_F = \varepsilon \, |i\rangle\langle i| \,, \tag{37}$$

where we write $|i\rangle$ to denote the fermion level and thus avoid confusion with the boson bound state denoted as $|1\rangle$.

ii.) A standard Friedrichs model as described in the previous section.

iii.) An interaction between the fermion level and the boson field given by the following reduced version of H_{III}:

$$H_{III} = \int d\omega \left[f(\omega)\, |i\rangle\langle i, \omega| + f^*(\omega)\, |i, \omega\rangle\langle i| \right]. \tag{38}$$

Thus, the total Hamiltonian is given by

$$H = \omega_0 \, |1\rangle\langle 1| + \int_0^\infty |\omega\rangle\langle\omega| \, d\omega + \varepsilon \, |i\rangle\langle i|$$

$$+ \lambda \int_0^\infty [V(\omega)|\omega\rangle\langle 1| + V^*(\omega)|1\rangle\langle\omega|] \, d\omega$$

$$+ \int d\omega \, [f(\omega) \, |i\rangle\langle i, \omega| + f^*(\omega) \, |i, \omega\rangle\langle i|] \,. \tag{39}$$

The interaction produced by H is depicted in Figure 1.

Take now the part of the Hamiltonian that corresponds to the standard Friedrichs model, given by $H_0 + \lambda V$ as in (1) and (2). It has been shown ([25]) that this Hamiltonian can be diagonalized in terms of the Gamow vectors and the background part as

$$H = z_R \, |f_0\rangle\langle\widetilde{f_0}| + \int_\Gamma z|z\rangle\langle z| \, d\mu(z) \tag{40}$$

In the description given by (40), the boson bound state has been replaced by the Gamow vector $|f_0\rangle$ and the boson field by the background. We can ignore or neglect the background term and then consider a two level system: one level is given by the fermion bound state $|i\rangle$ and the other by the Gamow vector $|f_0\rangle$. The resulting model can be seen as depicted in Figure 2.

The situation now as visualized in Figure 2 contains two systems in interaction for which the Hamiltonian can be written as[3]

$$H = \varepsilon|i\rangle\langle i| + z_R|f_0\rangle\langle\widetilde{f_0}| + \Lambda(z_R)[|i, f_0\rangle\langle i| + |i\rangle\langle i, \widetilde{f_0}|] \,, \tag{41}$$

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[3]As a matter of fact, this Hamiltonian should be written as

$$H = \varepsilon|i\rangle\langle i| \otimes I + z_R \, I \otimes |f_0\rangle\langle\widetilde{f_0}| + \Lambda(z_R)[|i, f_0\rangle\langle i| + |i\rangle\langle i, \widetilde{f_0}|] \,,$$

where I is the identity operator. In terms of creation and annihilation operators, this Hamiltonian can be written as

$$H = \varepsilon c_i^\dagger c_i + z_R G^\dagger G + \Lambda(z_R)[c_i^\dagger G^\dagger c_i + c_i^\dagger G c_i] \,,$$

where $c_i^\dagger$ and c_i are the creation and annihilation operators for the fermion with state $|i\rangle$ and $G^\dagger$ and G are the creation and annihilation operators for the Gamow state $|f_0\rangle$.

where the symbol $\Lambda(z_R)$ denotes a complex number which depends on the position of the resonance (and therefore on the value of the coupling constant λ) only.

It is important to remark that in (41) we have neglected the contribution of the background integral, (a term of the form $\int_\Gamma z|z\rangle\langle\widetilde{z}|\,d\mu(z)$) and note that consequently the resulting Hamiltonian is not formally self-adjoint.

Our goal is to show the existence of resonances, i.e., complex solutions in η of the eigenvalue equation

$$H\varphi = \eta\varphi\,.$$

The most general form of the vector φ can be given as:

$$\varphi = \alpha|i,0\rangle + \beta|i,f_0\rangle\,, \tag{42}$$

where: i.) The vector $|i,f_0\rangle$ is the tensor product $|i\rangle \otimes |f_0\rangle$. ii.) The vector $|i,0\rangle$ is the tensor product $|i\rangle \otimes |0\rangle$, where $|0\rangle$ denotes a state with zero resonances. In the basis given by $\{|i,0\rangle, |i,f_0\rangle\}$ the matrix elements of H can be readily obtained:

$$H = \begin{pmatrix} \langle i,0|H|i,0\rangle & \langle i,0|H|i,f_0\rangle \\ \langle i,\widetilde{f_0}|H|i,0\rangle & \langle i,\widetilde{f_0}|H|i,f_0\rangle \end{pmatrix} = \begin{pmatrix} \varepsilon & \Lambda(z_R) \\ \Lambda(z_R) & \varepsilon + z_R \end{pmatrix}\,. \tag{43}$$

In order to obtain the eigenvalues of H, we need to solve the equation $\det(H - E) = 0$ and this gives:

$$\det(H - E) = (\varepsilon - E)(\varepsilon + z_R - E) - \Lambda^2(z_R) = 0\,, \tag{44}$$

i.e.,

$$E^2 - (2\epsilon + z_R)E + z_R\epsilon - \Lambda^2(z_R) = 0\,, \tag{45}$$

which gives

$$E = \varepsilon + \frac{z_R}{2} \pm \frac{1}{2}\sqrt{z_R^2 + \Lambda^2(z_R)}\,. \tag{46}$$

Observe that, for any given resonance z_R in the boson levels, there exists two new resonances in the coupled boson-fermion system.

3.1 Calculation of $\Lambda(z_R)$.

In the simplified version under our consideration, only the interaction Hamiltonians H_{III} and H_{IV} remain. For the former, we keep only one term in the sum in (22) so that it has now the form:

$$H_{III} = \int_0^\infty [f(\omega)|i,0\rangle\langle i,\omega| + f^*(\omega)|i,\omega\rangle\langle i,0|\,] \, d\omega \,. \tag{47}$$

Concerning H_{VI}, it only survives a term of the sum in (23) and the corresponding form factor (the $g(\omega)$) is the form factor of the boson-boson field coupling $V(\omega)$. As it was previously clarified, this coupling is the responsable for the creation of the boson resonance given by $|f_0\rangle$.

We now want to determine the number[4] $\Lambda(z_R)$ in the approximation that neglects the background. Note that (41) is, save for the background term equal to $H_I + H_{IV} + H_{III}$, with only one fermion bound term in the fermion sum in H_I. Since $\varepsilon|i\rangle\langle i| + z_R|f_0\rangle\langle\widetilde{f_0}| + \text{background} = H_I + H_{IV}$, we have that

$$H_{III} = \Lambda(z_R)[|i,f_0\rangle\langle i,0| + |i,0\rangle\langle i,\widetilde{f_0}|\,] \,. \tag{48}$$

If we multiply (48) to the right by the ket $|i,f_0\rangle$, and take into account that $\langle i,0|i,f_0\rangle = 0$ and $\langle i,\widetilde{f_0}|i,f_0\rangle = \langle i|i\rangle\langle\widetilde{f_0}|f_0\rangle = 1$ [25, 15], we obtain:

$$\int f(\omega)|i\rangle\langle i,\omega|i,f_0\rangle \, d\omega = \left[\int_0^\infty f(\omega)\langle\omega|f_0\rangle \, d\omega\right]|i\rangle = [\Lambda(z_R)]\,|i\rangle\,, \tag{49}$$

which obviously gives:

$$\Lambda(z_R) = \int_0^\infty f(\omega)\langle\omega|f_0\rangle \, d\omega \,. \tag{50}$$

The value for the bracket $\langle\omega|f_0\rangle$ is readily obtained from (18). This gives:

$$\Lambda(z_R) = \lambda \int_0^\infty \frac{f(\omega)\,V(\omega)}{z_R - \omega + i0} \, d\omega$$

$$= \lambda\,\mathrm{PV} \int_0^\infty \frac{f(\omega)\,V(\omega)}{z_R - \omega} \, d\omega - i\pi\lambda f(z_R)\,V(z_R)\,. \tag{51}$$

---

[4]We write it as a function of the position of the resonance pole, although we equally can write it as a function of the value of the coupling constant λ which determines the pole, i.e., as $\Lambda(\lambda)$.

Here, PV stands for principal value (although since z_R is complex the principal value is here the value of the integral). We are always assuming that both form factors, $f(\omega)$ and $V(\omega)$, are analytically continuable and that have a certain well defined value at z_R.

Neglecting the integral term in (51) is an approximation similar to neglecting the background term. From this point of view, we arrive at

$$\Lambda(z_R) \approx -i\pi\lambda f(z_R)V(z_R)\,. \tag{52}$$

Note that, after (52), $\Lambda(z_R)$ is complex.

3.2 The effect of multiple fermion levels.

We are going to introduce a generalization to the above discussed simplified model. Since we are discussing decay in nuclei, it seems natural to add multiple levels for the fermion bound states. Now, in the extended Friedrichs model we keep the following terms:

1.- The free Hamiltonian H_I as in (20).

2.- The interaction between fermions and the boson H_{II} as in (21).

3.- The interaction between fermions and the boson field H_{III} given by (22).

4.- The interaction boson, boson field given by (2). Again, note that this is a simplified form of H_{IV}.

The strategy is the same as in the previous subsection: we replace the boson-boson field by the resonance-background and we neglect the background field keeping the Gamow state. Then, we have an interaction between the bound state fermions and the boson resonance given by[5]

$$H_{\text{int}} = \sum_{kl} \Lambda_{kl}(z_R)[|k, f_0\rangle\langle l, 0| + |k, 0\rangle\langle k, \widetilde{f_0}|]\,. \tag{53}$$

---

[5]If $c_k^\dagger$ and c_k respectively are the creation and annihilation operators for the $k-th$ fermion and $G^\dagger$ and G the creation and annihilation of the Gamow, then, this interaction Hamilonian can be written in terms of these operators as:

$$H_{\text{int}} = \sum_{kl} \Lambda_{kl} c_k^\dagger c_k (G^\dagger + G)\,.$$

We can operate either in the language of vectors, as shall do in the main text or in the language of creation and annihilation operators as is usual in the second quantization formalism.

Repeating the procedure of the previous subsection, we identify the interaction Hamiltonian, excluding λV, with H_{int} given in (53). This gives the following identity:

$$H_{\text{int}} = \sum_{kl}[h_{kl}|l,0\rangle\langle k,1| + h_{kl}^*|k,1\rangle\langle l,0|\,]$$

$$+ \sum_{kl}\int_0^\infty [f_{kl}(\omega)|l,0\rangle\langle k,\omega| + f_{kl}^*(\omega)|k,\omega\rangle\langle l,0|\,]\,. \tag{54}$$

Multiplying (54) to the right by $|m,f_0\rangle$, we obtain:

$$\sum_{kl}h_{kl}|l,0\rangle\langle k,1|m,f_0\rangle + \sum_{kl}\int_0^\infty f_{kl}(\omega)|l,0\rangle\langle k,\omega|m,f_0\rangle\,d\omega$$

$$= \sum_{kl}\Lambda_{kl}|l,0\rangle\langle k,\widetilde{f_0}|m,f_0\rangle\,. \tag{55}$$

Since $\langle 1|f_0\rangle = 1$ and $\langle k|m\rangle = \delta_{km}$, (55) yields to

$$\sum_l\{h_{ml} + \int_0^\infty f_{ml}(\omega)\langle\omega|f_0\rangle\,d\omega\}|l,0\rangle = \sum_l\Lambda_{ml}(z_0)|l,0\rangle\,. \tag{56}$$

As the $|l,0\rangle$ are linearly independent, (56) gives for all m,l:

$$\Lambda_{ml}(z_0) = h_{ml} + \int_0^\infty f_{ml}(\omega)\langle\omega|f_0\rangle\,d\omega$$

$$\approx h_{ml} - i\pi f_{ml}(z_0)\lambda V(z_0)\,, \tag{57}$$

where again we have used (18) for the determination of the value of $\langle\omega|f_0\rangle$ and have neglected the term principal value as in (52).

The space of pure states is formed by the vectors of the form[6]

---

[6]It we use the second quantization language, the vector can be written as $\varphi(E) = \sum_k \alpha_k c_k^\dagger|0\rangle + \sum \beta_k c_k^\dagger G^\dagger|0\rangle$, where $|0\rangle$ represents the state of the vacuum.

$$\varphi(E) = \sum_k \alpha_k |k, 0\rangle + \sum \beta_k |k, f_0\rangle. \tag{58}$$

Then, we want to find the explicit form of the Hamiltonian in terms of the basis given by the $|k, 0\rangle$ and the $|k, f_0\rangle$ and then obtaining the eigenvalues of this Hamiltonian. The presence of complex eigenvalues of this Hamiltonian will be equivalent to the presence of resonances for the model.

In order to show in detail the calculation, we split the Hamiltonian into three terms as follows:

$$H_1 = \sum_k \varepsilon |k, 0\rangle\langle k, 0|, \tag{59}$$

$$H_2 = z_R |0, f_0\rangle\langle 0, \widetilde{f_0}|, \tag{60}$$

$$H_3 = \sum_{kl} \Lambda_{kl} |k, f_0\rangle\langle l, \widetilde{f_0}| + H_{\text{BG}}, \tag{61}$$

where H_{BG} denotes the background part of the Hamiltonian that we neglect and we have written Λ_{kl} instead of $\Lambda_{kl}(z_0)$ in order to simplify the notation. After simple manipulations, we obtain the following[7]:

$$\langle \varphi | H_1 | \varphi \rangle = \sum_{k'} \varepsilon_{k'} \{\alpha_{k'}^* \alpha_{k'} + \beta_{k'}^* \beta_{k'}\},$$

$$\langle \varphi | H_2 | \varphi \rangle = \sum_{k'} \{\beta_{k'}^* \beta_{k'}\} z_R,$$

$$\langle \varphi | H_3 | \varphi \rangle = \sum_{k'} \alpha_{k'}^* \Lambda_{k'}(\beta) + \sum_{k'} \beta_{k'}^* \Lambda_{k'}(\alpha), \tag{62}$$

where

---

[7]We arrive to the same results using second quantization notation. In this notation, $H_1 = \sum_k \varepsilon_k c_k^\dagger c_k$, $H_2 = z_R G^\dagger G$ and $H_3 = \sum_{kl} \Lambda_{kl}(G^\dagger + G)c_k^\dagger c_l$. We use the anti-commutation relation $\{c_k, c_l^\dagger\} = \delta_{kl}$ and the commutation relation $[G, G^\dagger] = 1$ and note that the action of $C^\dagger$ and $G^\dagger$ in the vacuum $|0\rangle$ is given by $c^\dagger |0\rangle = |k, 0\rangle$ and $G^\dagger |0\rangle = |0, f_0\rangle$.

$$\Lambda_{k'}(\alpha) = \sum_{k''} \Lambda_{k'k''}\alpha_{k''}, \quad \Lambda_{k'}(\beta) = \sum_{k''} \Lambda_{k'k''}\beta_{k''}. \qquad (63)$$

Using (62) and (43), we finally get:

$$\langle\varphi|H|\varphi\rangle = \sum_{k'}|\alpha_{k'}|^2\varepsilon_{k'} + \sum_{k'}|\beta_{k'}|^2(\varepsilon_{k'} + z_R) + \sum_{kk'}(\alpha_{k'}^*\beta_k + \beta_{k'}^*\alpha_k). \quad (64)$$

In order to obtain solutions for the eigenvalue equation $H\varphi = E\varphi$, we use the following variational principle:

$$\frac{\partial}{\partial\alpha_{k'}^*}\{\langle\varphi|H|\varphi\rangle - E\langle\varphi|\varphi\rangle\} = 0 \qquad (65)$$

$$\frac{\partial}{\partial\beta_{k'}^*}\{\langle\varphi|H|\varphi\rangle - E\langle\varphi|\varphi\rangle\} = 0. \qquad (66)$$

This gives a system of linear equations in the undetermined $\alpha_{k'}$, $\beta_{k'}$, which gives the eigenvector $|\varphi\rangle$ and E. This system is (we are assuming that $\Lambda_{kk'} = \Lambda_{k'k}$)

$$\alpha_{k'}\varepsilon_{k'} + \sum_{k'}\Lambda_{kk'}\beta_{k'} = E\alpha_{k'} \qquad (67)$$

$$\beta_{k'}(\varepsilon_{k'} + z_R) + \sum_{k}\Lambda_{kk'}\alpha_k = E\beta_{k'}. \qquad (68)$$

This can be also written as:

$$\alpha_{k'}(\varepsilon_{k'} - E) = -\sum_{k}\Lambda_{kk'}\beta_k, \qquad (69)$$

$$\beta_{k'}(\varepsilon_{k'} + z_R - E) = -\sum_{k}\Lambda_{kk'}\alpha_k. \qquad (70)$$

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Thus, the eigenvalue equation can be written in the following matrix form:

$$
\begin{pmatrix} \mathcal{E} & \mathbf{\Lambda} \\ \mathbf{\Lambda} & \mathcal{E} + \mathcal{Z} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = E \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \tag{71}
$$

with

$$
\mathbf{\Lambda} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1k_n} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2k_n} \\ \cdots & \cdots & \cdots & \cdots \\ \Lambda_{k_n 1} & \Lambda_{k_n 2} & \cdots & \Lambda_{k_n k_n} \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} \varepsilon_{k_1} & 0 & \cdots & 0 \\ 0 & \varepsilon_{k_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_{k_n} \end{pmatrix} \tag{72}
$$

$$
\mathcal{E} + \mathcal{Z} = \begin{pmatrix} \varepsilon_{k_1} + z_R & 0 & \cdots & 0 \\ 0 & \varepsilon_{k_2} + z_R & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_{k_n} + z_R \end{pmatrix} \tag{73}
$$

$$
\mathbf{a} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{k_n} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{k_n} \end{pmatrix}, \tag{74}
$$

where k_n is the total number of fermions considered.

If we have one fermion state, only, we are in the case studied in the previous subsection. If we have two fermions states, the eigenvalue equation is a equation of fourth order and therefore solvable. In the simplest case in which

$$
\mathbf{\Lambda} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \tag{75}
$$

a being a complex constant, the eigenvalues of equation (71) fulfill the following equation:

$$
\det \begin{pmatrix} \varepsilon_1 - E & 0 & 0 & a \\ 0 & \varepsilon_2 - E & a & 0 \\ 0 & a & \varepsilon_1 + z_R - E & 0 \\ a & 0 & 0 & \varepsilon_2 + z_R - E \end{pmatrix} = 0
$$

$$
= \{(\varepsilon_1 - E)(\varepsilon_2 + z_R - E) - a^2\}\{(\varepsilon_2 - E)(\varepsilon_1 + z_R - E) - a^2\}. \tag{76}
$$

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The solutions of (76) are

$$E_{1,2} = \frac{1}{2}\left(z_R + \varepsilon_1 + \varepsilon_2 \pm \sqrt{(z_R + \varepsilon_1 + \varepsilon_2)^2 - 4(\varepsilon_1\varepsilon_2 + \varepsilon_2 z_R - a^2)}\right) \quad (77)$$

$$E_{3,4} = \frac{1}{2}\left(z_R + \varepsilon_1 + \varepsilon_2 \pm \sqrt{(z_R + \varepsilon_1 + \varepsilon_2)^2 - 4(\varepsilon_1\varepsilon_2 + \varepsilon_1 z_R - a^2)}\right). \quad (78)$$

Obviously, these four solutions are complex.

3.3 Realistic Applications

In this subsection we shall rephrase some results of the previous section in terms of the particle-vibration coupling scheme [4] and, particularly, in the context of the particle-vibration coupling in Bergreen's basis [18].

To start with, we shall discuss the structure of the particle-vibration coupling, in a resonant basis and by using separable two-body interactions. The boson sector of the Hamiltonian is usually treated in terms of the TDA (Tamm-Damcoff) approximation, leading to vertex functions of the type [18]

$$\Lambda(\omega, \eta) = \sum_\nu \frac{V_{\eta,\nu}^2}{(e_\nu - \omega)} \quad (79)$$

where V is the matrix element of the two-body interaction, ν and η denote particle-hole configurations and ω is the energy of the one phonon state. If in the configuration η (or ν) a single-particle resonant state participates, the energy of the one phonon state, ω, becomes complex and the imaginary part of this energy will be proportional to the imaginary part of the energy of the particle-hole configuration. Tipically, in a large single particle basis, with few resonant states, the imaginary part of the energy of a high-lying one-phonon state is of the order of 1 keV or smaller [18].

In the context of the results of the previous subsection, there is a direct correspondence between Eq. 51 and $\Lambda(\omega, \eta)$. Note that, while $\Lambda(z_R)$ has been obtained by working with the fermion-boson interaction firstly, the result $\Lambda(\omega, \eta)$ has been obtained (see [18]) by diagonalizing the boson sector and then by using it in the transformation of the particle-vibration coupling.

The just mentioned analogy suggest that both procedures may be equivalent. These procedures are:

i.) The treatment of particle-vibration couplings in resonant (Bergreen) basis. It consists in including fermion resonant states in single-particle basis, solving for TDA or RPA equations and getting complex phonon energies, extracting vertex functions and performing a particle-vibration coupling calculation between effective fermion and boson fields.

ii.) The extended Friedrichs model: It consists in solving the Friedrichs model equations proposed in this work, isolating a boson resonant state, ignoring the background and performing a particle-vibration coupling scheme by using bound fermions and the resonant boson.

In the case of the simplified versions of section 3.2, the solution E_4, in (78), closely resembles some of the results of [18]. For the case of one particle above the Fermi level E_4 yields a physical value

$$E_4 \approx \varepsilon_1 - \Lambda^2(z_R) \tag{80}$$

where we have set $a = \Lambda(z_R)$. Thus, the coupling with the resonant boson state leads to a imaginary energy of the order of the imaginary part of the quantity $\Lambda(z_R)$.

4 Conclusions

In this work we have presented an extended version of the Friedrichs model. It consists of couplings between fermion and bosons, in addition to the couplings between the discrete and continuous boson excitations. The extended model is exactly solvable and the structure of the solutions shows a suitable mechanism for the appearance of complex energies in the fermion spectrum. In order to show the possible relevance of the present results for nuclear structure physics we have performed a comparison with the results obtained in realistic coupling schemes. The coincidence of the formal structure of the solutions, for fermion states, in both approaches, suggests a direct conection between them. The results allow for the interpretation of the imaginary part of the energy of the fermion states obtained by coupling fermion bound and resonant states to collective nuclear particle-hole excitations. Further studies, concerning the calculation of two-particle correlation functions and particle-phonon correlation functions in resonant basis are in progress.

5 Acknowledgements

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