Nonstandard q-deformed realizations of the harmonic oscillator

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The boson expansion method is applied to find the spectrum of a q-deformed harmonic oscillator. We use two different boson expansions, each of them including a deformation parameter, defined in terms of exponential and logarithmic functionals. The resulting Hamiltonians are bilinear forms of the transformed operators. Physical effects resulting from the deformation of the generators of the algebra are studied by comparing known finite-range potentials and the effective potentials obtained for each of the considered Hamiltonians.

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I. INTRODUCTION

Mathematical aspects of the so-called quantum-deformed (q-deformed) algebras have been studied intensively in the past [1,2]. The search for physically inspired Hamiltonians, which may display definite features about q-deformation effects, is still open. Concrete applications of the formalism have been explored more recently [3–8] and these complement previous mathematical efforts, such as studies of generalized q-deformed oscillators [9,10]. In a series of papers, Wess and coworkers [11–15] and Zhang and coworkers [16–18] have studied different nonstandard q-deformation schemes and obtained the corresponding Hamiltonians. Previously, we have applied these ideas to study fermion-boson couplings [19], and the possible correspondence between q-deformations and boundary conditions [20], by comparing the spectrum of a q-deformed harmonic oscillator with that of a finite-range potential.

In this work, we continue the study presented in [20], by applying nonstandard realizations of quantum deformed algebras [21–28], in conjunction with boson expansions. We have calculated the eigenvalues of Hamiltonians that include bilinear and quadratic forms of the generators of the algebra. We have chosen exponential and logarithmic functionals of the deformation parameter and of the bosons to deform the generators of the algebra. The comparison between different realizations is done by diagonalizing the corresponding Hamiltonians in the basis of states generated by the bosons introduced by the mapping procedure. Also, we have constructed, for each of the considered Hamiltonioans, effective potentials. This was done by applying to the Hamiltonians suitable gauge transformations, as was explained in [20].

In Sec. II we present the nonstandard realizations we have used, as well as the corresponding Hamiltonians. Numerical results are presented in Sec. III. Conclusions are drawn in Sec. IV.

II. FORMALISM

A. Nonstandard q deformations

To fix the concept of a deformed algebra [25], let us introduce the commutation relations between the generators N, A_+, A_- of a standard oscillator algebra:

$$[N, A_\pm] = \pm A_\pm, \qquad [A_-, A_+] = 1. \tag{1}$$

If λ is taken as a *deformation* parameter, one may introduce the following commutation relations:

$$\begin{aligned}[N, A_+] &= \frac{e^{\lambda A_+} - 1}{\lambda},\\ [N, A_-] &= -A_-,\\ [A_-, A_+] &= e^{\lambda A_+},\end{aligned} \tag{2}$$

which in the limit $\lambda \to 0$ yields Eqs. (1). This set of equations (2) is an example of a *nonstandard* deformed algebra [21–28], in contrast to the *standard* one [6–8], which can be defined by the following commutation relations:

$$\begin{aligned}[N, A_\pm] &= \pm A_\pm,\\ [A_-, A_+]_q &= q^N.\end{aligned} \tag{3}$$

The q-deformed Casimir operator C_λ, corresponding to the nonstandard algebra (2), is defined by

$$C_\lambda = N - \frac{1}{2}\left\{\frac{1 - e^{-\lambda A_+}}{\lambda} A_- + A_- \frac{1 - e^{-\lambda A_+}}{\lambda}\right\}. \tag{4}$$

In the limit $\lambda = 0$ it reduces to the form

$$C = N - \tfrac{1}{2}\{A_+, A_-\}, \tag{5}$$

or, in a more familiar way,

$$N = \tfrac{1}{2}\{A_+, A_-\} + C. \tag{6}$$

These are the elements needed to complete the definition of a nonstandard q-deformed algebra and we shall use them to construct a boson picture of the involved operators.

B. Boson mapping

Let us introduce the boson creation and annihilation operators $a^\dagger$ and a, respectively. Then, following the techniques of [29], we can construct the boson pictures of the generators

A_+, A_-, and N,

$$\begin{aligned} A_+ &= a^\dagger, \\ A_- &= a, \\ N &= a^\dagger a, \end{aligned} \tag{7}$$

where $[a, a^\dagger] = 1$. This mapping can be deformed, as we are going to show next, by replacing the right-hand side of each of the equalities of (7) by functionals of the boson operators weighted by the deformation parameter λ. Among the possible realizations we have selected exponential and logarithmic forms.

1. q-deformed exponential boson mapping

The exponential boson mapping is defined as

$$\begin{aligned} A_+ &= a^\dagger, \\ A_- &= e^{\lambda a^\dagger} a, \\ N &= \frac{e^{\lambda a^\dagger} - 1}{\lambda} a, \end{aligned} \tag{8}$$

which reduces to (7) for $\lambda = 0$.

2. q-deformed logarithmic boson mapping

Another boson mapping is the logarithmic transformation

$$\begin{aligned} A_+ &= \frac{1}{\lambda} \ln\left(\frac{1}{1 - \lambda a^\dagger}\right), \\ A_- &= a, \\ N &= a^\dagger a, \end{aligned} \tag{9}$$

which also reduces to (7) in the limit $\lambda = 0$.

C. q-deformed Hamiltonians

In this subsection we analyze the structure of the q-deformed boson operators, both for the exponential and logarithmic mappings, by expanding them as power series of the deformation parameter λ. Next, we transform symmetric and asymmetric quadratic Hamiltonians to these q-deformed boson basis. This will allow us to establish a connection between q-deformed Hamiltonians and the ones we have chosen for comparison, namely, the Woods-Saxon (WS) and Poeschl-Teller (PT) potentials. We shall perform an analysis similar to the one presented in our previous work [20].

The procedure we have adopted consists of the following steps:

(a) the expansion of the q-deformed boson operators at leading order in λ;

(b) the use of a gauge transformation, constructed with the deformed boson operators, to write the eigenvalue equation for the Hamiltonians.

Step (b) will allow us to write the associated Schrödinger equations, and, in turn, to extract the potentials induced by the deformation of the algebra. At this point we shall then be able to compare the obtained potentials with the WS and PT ones.

1. q-deformed generalized momentum and coordinate operators

The momentum and coordinate operators can be written [29] as linear combinations of the operators A_+ and A_-, as is usually done in the harmonic oscillator representation, namely,

$$\begin{aligned} P &= i\sqrt{\frac{m\omega\hbar}{2}}(A_+ - A_-), \\ X &= \sqrt{\frac{\hbar}{2m\omega}}(A_+ + A_-). \end{aligned} \tag{10}$$

Both expressions can be transformed to the boson basis, using the boson mappings (8) and (9), leading to the equations

$$\begin{aligned} P &= p - im\omega\Theta_\lambda, \\ X &= x + \Theta_\lambda, \end{aligned} \tag{11}$$

with

$$\begin{aligned} p &= i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a), \\ x &= \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \end{aligned} \tag{12}$$

and

$$\Theta_\lambda = \sqrt{\frac{\hbar}{2m\omega}} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} a^{\dagger k} a \tag{13}$$

for the exponential form of the boson mapping (8) and

$$\begin{aligned} P &= p + im\omega\Phi_\lambda, \\ X &= x + \Phi_\lambda, \end{aligned} \tag{14}$$

where

$$\Phi_\lambda = \sqrt{\frac{\hbar}{2m\omega}} \sum_{k=2}^{\infty} \frac{\lambda^{k-1}}{k} a^{\dagger k}, \tag{15}$$

for the logarithmic form (9).

With these definitions, we can construct the q-deformed boson pictures of the Hamiltonian

$$\begin{aligned} H &= \frac{P^2}{2m} \\ &= \frac{\hbar\omega}{4}(A_+A_- + A_-A_+ - A_+^2 - A_-^2). \end{aligned} \tag{16}$$

This Hamiltonian, using the mapping (8), has the boson image

$$H_{\text{exp}} = \frac{p^2}{2m} - \frac{1}{2}m\omega^2\Theta_\lambda^2 - i\frac{\omega}{2}\{p, \Theta_\lambda\}, \tag{17}$$

and, for the logarithmic mapping (9), it has the form

$$H_{\text{log}} = \frac{p^2}{2m} - \frac{1}{2}m\omega^2\Phi_\lambda^2 + i\frac{\omega}{2}\{p, \Phi_\lambda\}. \tag{18}$$

The Hamiltonian

$$H' = \frac{\hbar\omega}{4}[\eta A_+A_- + \zeta(A_+^2 + A_-^2)] \tag{19}$$

is a generalization of (16). The factors η and ζ have been introduced to allow for different situations, one of which is the pure harmonic oscillator $\hbar\omega a^\dagger a$ (for $\eta = 4$, $\zeta = 0$). The q-deformed boson images of H' are written as

$$\begin{aligned} H'_{\rm exp} &= f_1(\eta,\zeta)\frac{p^2}{2m} + \zeta\frac{1}{2}m\omega^2\Theta_\lambda^2 - i f_1(\eta,\zeta)\frac{\omega}{2}\{p,\Theta_\lambda\} \\ &\quad + f_2(\eta,\zeta)\frac{m\omega^2}{2}(x^2 + \{x,\Theta_\lambda\}) \\ &\quad - \frac{\omega\eta}{8}(\hbar - m\omega[x,\Theta_\lambda] + i[p,\Theta_\lambda]) \end{aligned} \tag{20}$$

and

$$\begin{aligned} H'_{\rm log} &= f_1(\eta,\zeta)\frac{p^2}{2m} + \zeta\frac{1}{2}m\omega^2\Phi_\lambda^2 + i f_1(\eta,\zeta)\frac{\omega}{2}\{p,\Phi_\lambda\} \\ &\quad + f_2(\eta,\zeta)\frac{m\omega^2}{2}(x^2 + \{x,\Phi_\lambda\}) \\ &\quad - \frac{\omega\eta}{8}(\hbar + m\omega[x,\Phi_\lambda] + i[p,\Phi_\lambda]), \end{aligned} \tag{21}$$

for the exponential and logarithmic mappings, respectively. The factors $f_1 = \frac{\eta-2\zeta}{4}$ and $f_2 = \frac{\eta+2\zeta}{4}$ of these equations reduce to $f_1 = 1$ and $f_2 = 0$, for $\eta = 2$ and $\zeta = -1$, for which (16) and (19) coincide, up to a term of the form $\frac{\hbar\omega}{4}\eta e^{\lambda A_+}$, since $A_-A_+ = A_+A_- + e^{\lambda A_+}$[see (2)]. Notice that the last term of both (20) and (21) appears because we have written A_+A_- instead of $\{A_-, A_+\}$ in (19).

D. Leading order expressions

In this section we analyze the structure of the q-deformed boson operators, both for the exponential and logarithmic mappings, by expanding them at leading order in the deformation parameter λ. The resulting expressions may then allow us to compare the effects produced on the Hamiltonians by the deformation of the algebra, which we would like to describe as added potentials.

1. Exponential mapping

To leading order in λ the operators of Eq. (8) are written as

$$\begin{aligned} A_+ &= a^\dagger, \\ A_- &\approx a + \lambda a^\dagger a + \frac{\lambda^2}{2}a^{\dagger 2}a. \end{aligned} \tag{22}$$

By keeping terms of the order λ^2, the Hamiltonian (16) reads

$$\begin{aligned} H_{\rm exp} &\approx \frac{\hbar\omega}{2}\left\{p^2 + \frac{\lambda}{\sqrt{2}}[ip + q - ip(p^2+q^2)]\right. \\ &\quad + \frac{\lambda^2}{2}\left[p^2 + \frac{5}{4}(p^2+q^2) - p^2q^2 - \frac{3}{4}p^4\right. \\ &\quad \left.\left. - \frac{1}{4}q^4 - i\frac{1}{2}p(p^2+q^2+2)q\right]\right\}. \end{aligned} \tag{23}$$

Notice that in these equations the position q and the momentum p are dimensionless variables.

At this point we may establish the relationship between the deformed Hamiltonian (23) and equivalent forms that originate in true interactions. This can be done by following the procedure of [20]. It is a straightforward procedure, albeit cumbersome from the point of view of the algebraic manipulations, which we shall avoid repeating here. The basic notion is to introduce a local gauge transformation, which depends only on position, to cast the Hamiltonian as a kinetic (second derivative) term plus potentials where the deformation parameters enter as a coupling strength.

The gauge transformation

$$\Psi(q) = e^{i\alpha(q)}\varphi(q), \tag{24}$$

with the local phase factor

$$\alpha(q) = -i\frac{\lambda}{6\sqrt{2}}q(3-q^2), \tag{25}$$

is used to transform the Hamiltonian (23). By keeping terms up to λ^2, the corresponding Schrödinger equation for the eigenfunction $\varphi(q)$ of $H_{\rm exp}$ is written as

$$-\frac{\hbar^2}{2m}\varphi''(q) + \frac{\hbar\omega}{16}\lambda^2(1-q^2)^2\varphi(q) = E\varphi(q). \tag{26}$$

This means that the deformation of the algebra is seen as a potential that contains a constant term and q^2 and q^4 terms. This form is indeed similar to the one corresponding to the PT potential, as was the case of our previous study [20].

For completeness we have applied the same procedure to the leading order expansion of (19). As in the previous case, we introduce a local gauge transformation (24), with

$$\alpha(q) = -i\frac{\lambda}{6\sqrt{2}}q\left(3\frac{(34\zeta-\eta)}{(2\zeta-\eta)} - q^2\right), \tag{27}$$

to obtain the equation of motion; the result is

$$\begin{aligned} &-\frac{\hbar^2}{8m}(\eta-2\zeta)\varphi''(q) + \frac{\hbar\omega}{4}\left(-\frac{\eta}{2} - \frac{(4\zeta-\eta)^2\lambda^2}{16(2\zeta-\eta)}\right. \\ &\quad - \frac{(\zeta+\eta)}{\sqrt{2}}\lambda q + \frac{q^2}{8}[4(\eta+\zeta) + \lambda^2(4\zeta-\eta)] \\ &\quad \left. + \frac{2\zeta+\eta}{2\sqrt{2}}\lambda q^3 - \frac{(2\zeta-\eta)}{16}\lambda^2 q^4\right)\varphi(q) = E\varphi(q). \end{aligned} \tag{28}$$

This time, the leading order differential form includes all powers (even and odd powers) up to q^4. Naturally, the interesting feature is now that odd powers of the position are obtained.

2. Logarithmic mapping

We proceed in a manner similar to that of the previous subsection, expanding the boson operators in the q-deformed logarithmic mapping. By keeping terms up to the order λ^2 one obtains

$$\begin{aligned} A_+ &\approx a^\dagger + \frac{\lambda}{2}a^{\dagger 2} + \frac{\lambda^2}{3}a^{\dagger 3}, \\ A_- &= a. \end{aligned} \tag{29}$$

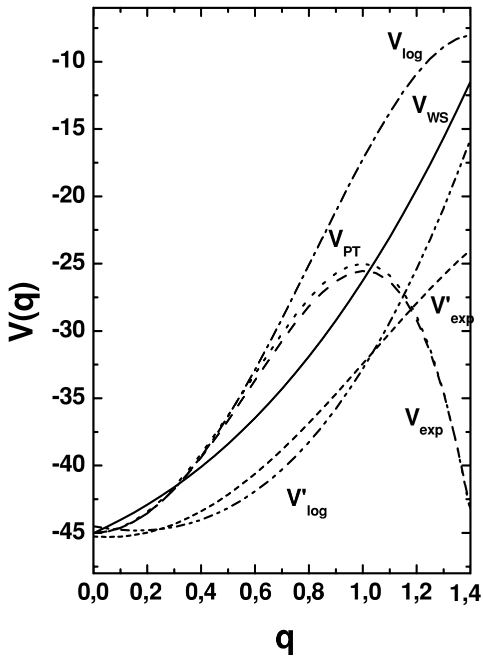

FIG. 1. The curves show the results for the potentials corresponding to the deformed Hamiltonians $H_{\rm exp}$(dotted line), $H_{\rm log}$(dashed-dotted line), $H'_{\rm exp}$ (short-dashed line), and $H'_{\rm log}$(dashed-dotted-dotted line) and for the PT (dashed line) and WS potentials (solid line). For the case of $H_{\rm exp}$, we have taken $\frac{\hbar\omega}{16}\lambda^2 = -20.0$; for $H_{\rm log}$, we have taken $\frac{\hbar\omega}{64}\lambda^2 = -9.0$. For the potentials associated with $H'_{\rm exp}$ and $H'_{\rm log}$, we have adopted the parameters $\hbar\omega = 20.0$, $\eta = 1.0$, and $\zeta = 1.1$. The parameters adopted for $V_{\rm PT}$ are $V_0 = -45$ and $R = 1.1$, whereas for $V_{\rm WS}$ we have taken $V_0 = -50.0$, $R = 1.1$, and $a_0 = 0.5$. The units are arbitrary units of energy, for $\hbar\omega$ and V_0, and length, for q, R, and a_0, respectively.

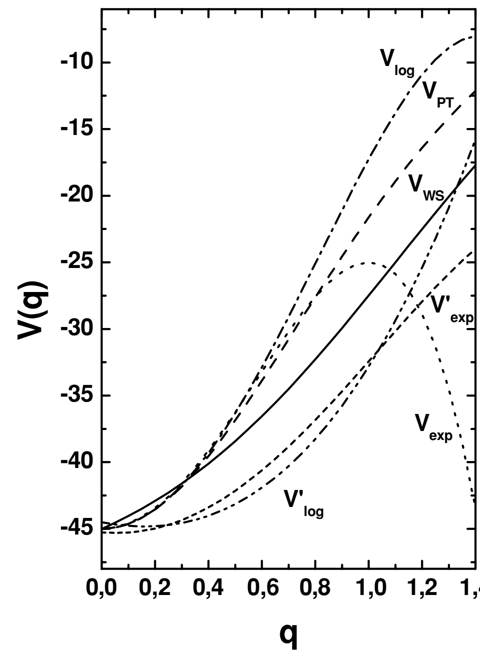

FIG. 2. The curves show the results for the potentials corresponding to the leading order expressions of the deformed Hamiltonians $H_{\rm exp}$(dotted line), $H_{\rm log}$(dashed-dotted line), $H'_{\rm exp}$ (short-dashed line), and $H'_{\rm log}$(dashed-dotted-dotted line) of Sec. II D and for the PT potential to order (q^6) (dashed line) and the WS potential to order (q^5) (solid line). The parameters adopted are those of Fig. 1.

This expansion yields, for the leading order terms of Hamiltonian (16) and for the gauge transformation

$$\alpha(q) = i\frac{\lambda}{12\sqrt{2}}q(6+q^2), \tag{30}$$

the effective potential

$$-\frac{\hbar^2}{2m}\varphi''(q) + \frac{\hbar\omega}{64}\lambda^2(2-q^2)^2\varphi(q) = E\varphi(q). \tag{31}$$

For the Hamiltonian (19), the expression in terms of the leading order contributions (29) and for the transformation with

$$\alpha(q) = -i\frac{(\eta+6\zeta)\lambda}{12\sqrt{2}(2\zeta-\eta)}q(3-q^2) \tag{32}$$

leads to the effective potential

$$\begin{aligned}&-\frac{\hbar^2}{8m}(\eta-2\zeta)\varphi''(q) + \frac{\hbar\omega}{4}\left\{-\frac{\eta}{2} - \frac{(6\zeta+\eta)^2}{64(2\zeta-\eta)}\lambda^2\right.\\ &-\frac{\eta}{2\sqrt{2}}\lambda q + \left[(\zeta + \frac{\eta}{2} + \frac{(6\zeta+\eta)^2}{32(2\zeta-\eta)}\lambda^2\right]q^2\\ &\left.+\frac{2\zeta+\eta}{4\sqrt{2}}\lambda q^3 - \frac{(6\zeta+\eta)^2}{64(2\zeta-\eta)}\lambda^2 q^4\right\}\varphi(q) = E\varphi(q).\end{aligned} \tag{33}$$

We may then conclude that the deformation of the algebra leads to effects that can be viewed as effective potentials. To compute them we need to calculate the matrix elements of the Hamiltonians we have considered in a given basis. This is done in the next subsection.

E. Matrix elements in the harmonic oscillator basis

The matrix elements of the Hamiltonian of Sec. II C are calculated in the harmonic oscillator basis; their expressions, in units of $\hbar\omega/4$, are

$$\begin{aligned}\langle l|H_{\rm exp}|n\rangle &= (2n+1)\delta(l,n) - \sqrt{n(n-1)}\delta(l,n-2)\\ &\quad - \sqrt{(n+1)(n+2)}\delta(l,n+2)\\ &\quad + \frac{(2n+1)\lambda^{l-n}}{(l-n)!}\sqrt{\frac{l!}{n!}}h(l-n)\\ &\quad - \frac{\lambda^{l-n+2}}{(l-n+2)!}\sqrt{\frac{l!}{n!}}\frac{n}{2}(n+l)\\ &\quad \times h(l-n+1),\end{aligned} \tag{34}$$

$$\begin{aligned}\langle l|H_{\rm log}|n\rangle &= (2n+1)\,\delta(l,n) - \sqrt{(n+1)(n+2)}\delta(l,n+2)\\ &\quad - \sqrt{(n)(n-1)}\delta(l,n-2)\\ &\quad - \sqrt{\frac{l!}{n!}}\lambda^{l-n}\left\{\lambda^{-2}\sum_{k=2}^{l-n-2}\frac{h(l-n-4)}{k(l-n-k)}\right.\end{aligned}$$

$$+ 2\lambda^{-2}\frac{h(l-n-3)}{(l-n-1)} - \left[\frac{l+n+3}{l-n+1}\right] \times h(l-n-1)\bigg\},\tag{35}$$

$$\begin{aligned}\langle l|H'_{\mathrm{exp}}|n\rangle = {}& \eta\frac{n\lambda^{l-n}}{(l-n)!}\sqrt{\frac{l!}{n!}}h(l-n)\\ &+ \zeta\sqrt{n(n-1)}\delta(l,n-2)\\ &+ \zeta\sqrt{(n+1)(n+2)}\delta(l,n+2)\\ &+ \zeta\frac{\lambda^{l-n+2}}{(l-n+2)!}\sqrt{\frac{l!}{n!}}\frac{n}{2}(n+l)h(l-n+1),\end{aligned}\tag{36}$$

and

$$\begin{aligned}\langle l|H'_{\mathrm{log}}|n\rangle = {}& \eta\sqrt{\frac{l!}{n!}}\frac{n\lambda^{l-n}}{(l-n+1)}h(l-n)\\ &+ \zeta\sqrt{(n)(n-1)}\,\delta(l,n-2)\\ &+ \zeta\sqrt{\frac{l!}{n!}}\lambda^{n-l-2}\sum_{k=1}^{l-n-1}\frac{h(l-n-2)}{k(l-n-k)},\end{aligned}\tag{37}$$

respectively. In these equations $h(x)$ is the step function $[h(x<0)=0]$. In the next section, we shall compare the solutions for definite values of the deformation parameter λ and for fixed values of the asymmetry parameters η and ζ.

III. RESULTS AND DISCUSSIONS

In this section we present the results of our calculations, which have been performed in the configuration space consisting of fifty harmonic oscillator shells. The general structure of the four Hamiltonians, H_{exp}, H_{log}, H'_{exp}, H'_{log}, consists of quadratic terms, p^2, q^2, Θ^2_λ, or Φ^2_λ, and extra terms that are proportional to product of p and q with Θ_λ and Φ_λ. Based on previous experience [20], one may expect that these extra terms would give rise to finite-range effects. To establish a comparison between q deformations and finite-range effects, for the nonstandard q-deformed realizations described here, we have chosen the WS and PT potentials as reference potentials [20]. The WS potential is a finite-range potential of the form

$$V_{\mathrm{WS}}(r) = \frac{V_0}{1+e^{\frac{r-R}{a_0}}},\tag{38}$$

whereas the PT potential is of the form

$$V_{\mathrm{PT}}(r) = \frac{V_0}{\left[\cosh\left(\frac{r}{R}\right)\right]^2}.\tag{39}$$

We may then compare the dependence of these potentials on position, with the corresponding ones of the Hamiltonians of Secs. II C and II D. In the calculation of the potentials of Sec. II C we have kept contributions up to the sixth power of q. The calculation of the leading order forms, of Sec. II D is restricted to the powers obtained in each case, after the application of the gauge transformations. The results are shown in Figs. 1 and 2.

The results shown in both figures support our claim about the effects induced by the deformation of the algebra on the harmonic oscillator. The effects may be represented by q^n potentials. The agreement between the WS and PT potentials with the one extracted from the q-deformed logarithmic boson expansion is rather acceptable (see Fig. 1) up to distances of the order of 1.4 times the size of the potentials. In Fig. 2, the leading order expressions of the deformed Hamiltonians of Sec. II D are compared to the truncated expressions of the WS and PT potentials, up to powers of the order of the sixth power of q, the dimensionless position variable. It is seen that the deformed Hamiltonians are better at reproducing the shape of the WS potential.

In parallel we have performed other calculations, by diagonalizing the expressions for the WS and PT potentials as well as the expression shown in Sec. II E, and we have observed that, depending on the adopted q-deformation scheme, the eigenvalues of the q-deformed harmonic oscillator may acquire complex eigenvalues. The imaginary part of these energies may also exhibit a revival pattern, in addition to the expected oscillatory behavior. However, the fact that the WS potential has only a finite number of bound states, with real energies, may not allow for a direct comparison of the eigenvalues, and for this reason we have restricted our comparison to the actual shape of the potentials. A direct comparison is indeed possible by setting up a cutoff energy.

IV. CONCLUSIONS

In this work, we have applied a nonstandard q deformation of the harmonic oscillator algebra to explore the features of the resulting spectrum. Boson images of the chosen Hamiltonians have been constructed. The calculations, which have been performed for various sets of parameters, show strong similarities between the shape of finite-range potentials, of the WS and PT type, and the shape of the q-deformed harmonic oscillator.

In conclusion, the present results may be used to show that the extraction of q-deformation parameters, starting from the spectrum of an effective Hamiltonian, is indeed feasible.

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