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Physical and mathematical aspects of Gamow states

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Abstract

Physical and mathematical aspects related to the description of resonant states are presented in a comprehensive way. The concepts concerning the representation of Gamow resonances are revisited in connection with a rigorous mathematical treatment, based on the use of Møller operators and rigged Hilbert spaces. The formalism is cast in a form amenable for applications to nuclear structure calculations in the continuum.

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1. Introduction

In spite of the long-time elapsed since the discovery of decay phenomena in nuclei and their description in terms of resonances, the use of the concept in nuclear structure calculations has been hampered by an apparent contradiction with conventional quantum mechanics, the so-called *probability problem*. It refers to the fact that a state with complex energy cannot be the eigenstate of a self-adjoint operator, like the Hamiltonian, therefore resonances are not vectors belonging to the conventional Hilbert space.

This report is devoted to the description of resonances, i.e. Gamow states, in an amenable mathematical formalism, i.e. Rigged Hilbert Spaces. Since we aim at further applications in the domain of nuclear structure and nuclear many-body problems, we shall address the issue in a physical oriented way, restricting the discussion of mathematical concepts to the needed, unavoidable, background. From a rigorous historical prospective, the sequence of events and papers leading to our modern view of Gamow vectors in nuclear structure physics includes the following series of publications: [17,103,111,74–76]. The use of Gamow states in conventional scattering and nuclear structure problems was advocated by Berggren, at Lund, in the 1960s [17] and lately [18], by Romo [103] and by Gyarmati and Vertse [111]. The notion was recovered years later, by Liotta, at Stockholm, in the 1980s [76], in connection with the microscopic description of nuclear giant resonances, alpha decay and cluster formation in nuclei. After these pioneering works, the use of Gamow states in nuclear structure calculations become widespread, in spite of the numerical and mathematical difficulties found in the implementation of calculations of realistic nuclear interactions in single-particle

resonant basis. Recently, both at GANIL-Oak Ridge [74] and at Stockholm [75], the use of Gamow states in shell model calculations was reported.

A parallel mathematical development took place, this time along the line proposed by Bohm and collaborators [20]. Curiously enough both attempts went practically unnoticed to each other for quite a long time until recently, when some of the mathematical and physical difficulties found in numerical applications of Gamow states were discussed on common grounds. The need to account for nuclear properties in the continuum, and the notion that the nuclear continuum can be replaced by few non-overlapping Gamow states in the so-called Berggren basis, i.e. single-particle basis which include bound single-particle states and few single-particle resonances and that can be used to perform approximate Tamm–Dancoff (TDA) or random phase approximation (RPA) diagonalization of residual nuclear two-body interactions [76], drove the attention of nuclear structure physicist to the more mathematically oriented work of Bohm and collaborators. However, to the best of our knowledge, a review of these approaches is still missed in the literature. We have taken the existence of this gap as the motivation for the present work.

Technically speaking, one may concentrate the pros and drawbacks of Berggren and Bohm approaches in the following:

(i) The formalism developed by Berggren is oriented, primarily, to the use of a mixed representation where scattering states and bound states are treated on a foot of equality. The normally accepted notion that the continuous should always be taken into account as a sort of complementary subspace of the space spanned by the bound states and resonances was revisited by Berggren. In his approach a *basis* should contain bound states and few resonances, namely those which have a small imaginary part of the energy. Thus, it is a certain degree of ambiguity in the choice of the so-called *narrow* resonances [103]. No further requirements are imposed, as a departure from, for instance, the infinite box-discretization method.

(ii) In Bohm's approach the steps towards the description of Gamow states are based on the fact that the spectrum of the analytically continued Hamiltonian covers the full real axis. A direct consequence of this is the definition, according to Bohm [20,42], of the expectation value of the Hamiltonian on a Gamow state. As a difference with respect to Berggren's method, the calculations in Bohm's approach are facilitated by the use of analytic functions.

The subjects included in this report can be ordered in two well-separated conceptual regions, namely: (a) the S-matrix theory, with reference to Møller wave operators, to the spectral theorem and to the basic notions about Rigged Hilbert spaces, and (b) the physical meaning of Gamow resonances in dealing with calculation of observables. The mathematical tools needed to cover part (a) are presented in a glossary of self-contained subsections. In that part of the report we shall follow, as closely as possible, the discussion advanced in the work of Arno Bohm and co-workers [20–25].

The conflict between the probabilistic interpretation of quantum mechanics and the description of resonances, Gamow states, and its solution in an amenable mathematical formalism, Rigged Hilbert spaces, is addressed to illustrate the suitability of the adopted representation. We think that this discussion is needed to pass a conceptual barrier built by some to argue against the use of Gamow states in nuclear structure calculations. In order to do it, we have addressed the link between mathematical and physical aspects of Gamow states, which we think is one of the lively topics in realistic nuclear structure calculations. In this respect we shall review, in a modern perspective, the work advanced by the late Tore Berggren [17,18].

Concerning the realization of the mathematical representation of Gamow states, we shall focus on the study of the Friedrichs model. This is a nice example of a solvable model where the use of generalized vectors in rigged Hilbert spaces becomes very simple and where the identification of Gamow states can be performed without difficulties.

As said before, in the first part of the report we focus on the elementary mathematical and physical concepts related to the use of Gamow states. Concerning part (b), we focus on the calculation of observables on resonant states and compare various prescriptions to calculate the mean value of the energy. For reasons of brevity we have avoided to include in the present work other specific examples on the use of Gamow states, but for this we relay mostly on the work done in specific nuclear structure calculations by Liotta and collaborators, at Stockholm and Debrecen. We shall refer to the work done at Stockholm [76], to show the usefulness of the approach based on the inclusion of Gamow states in nuclear single-particle basis, in the calculation of mean field properties and two-body interactions in nuclei. Particularly relevant for our present discussion are the results of the Stockholm group related to the calculation of the decay of nuclear resonances, like isobaric analogue states and Gamow–Teller resonances, as well as the exploration of mean field properties, like nuclear superconductivity, for nuclei in the drip line [76]. Another indication about the relevance of the use of Gamow states in nuclear structure calculations may be the recent announcement of the measurement of two-proton radioactivity.

The material is organized in the following way: Section 2 contains the mathematical ingredients of the problem, Section 3 is devoted to the discussion of physical concepts and properties of Gamow states, including the presentation and discussion of Friedrichs model. In Section 4 we address the problem of the mean value of the energy for resonance states. The extension of the formalism to the relativistic regime is discussed in Section 5. Conclusions are drawn in Section 6. Some additional mathematical details are presented in Appendices A and B. For the sake of convenience, the references have been ordered alphabetically, rather than chronologically.

2. Mathematical concepts

The notion of time evolution, either in the Heisenberg or in the Schrödinger representation, is a cornerstone in ordinary quantum mechanics. During the past century, since the discovery and formulation of quantum mechanics, the notion of time evolution has been exhaustively explored. However, this has not been the case with Gamow ideas on resonance phenomena, where the time evolution is determined by the presence of a complex parameter. Although the occurrence of this complex parameter does not present a serious formal difficulty, it is not possible to accommodate it in the framework of ordinary quantum mechanics. This may explain why this notion has been somehow neglected for a long period of time.

Generally speaking, we may call *resonance phenomena* those for which the conventional definition of probability are not applicable. Conventional quantum mechanics deals with two kinds of states: bound states and scattering states. Resonances can only appear together with scattering states.

Resonances are characterized by two real parameters: the resonance energy E_R and the width Γ. The quantity $2\hbar/\Gamma$, which is essentially the inverse of the width, represents the half life of the resonance. Therefore, a resonance can be viewed as a particle that lives a finite time $\Delta t \leqslant 2\hbar/\Gamma$.

From this point of view, bound states (representing stable particles, with $\Delta t = \infty$) and resonances (representing unstable particles, with $\Delta t =$ finite) differ solely in the value of their widths.

In order to understand resonance phenomena better, we want to discuss here the resonance scattering [20]. In a resonant scattering process a state of a particle is prepared in the remote past. It evolves freely until it enters in the interaction region, in which the particle remains a relatively large time, compared with the time scale of the non-interacting case. Then, it goes out of the interaction region and evolves freely until the state is detected in the far future, i.e., when it reaches the asymptotic limit. A relatively large time delay spent in the interaction region characterizes the existence of a resonance. This long-time delay reveals that a quasi-stationary state has been formed between the incoming particle and the center of forces, usually another particle which interacts with the former [20]. The scattering processes which we are considering are characterized by

(i) the free Hamiltonian H_0 governing the evolution of the state before and after the interaction takes place, and

(ii) the interaction V.

The total Hamiltonian is $H = H_0 + V$. In mathematical terms, we may speak of two Hamiltonians, $\{H_0, H\}$, that characterize the scattering process [1,100,114].

Whether a prepared state would give rise to a resonance in a scattering process, characterized by the Hamiltonians $\{H_0, H\}$, depends on the potential $V = H - H_0$ and on the energy of the incident particle. The time delay would appear only at certain discrete values the energy, which are called *resonant energies*. We shall elaborate these ideas in the next sections.

The most relevant concept in the description of a scattering process is the S-operator, or the S-matrix. This operator relates the prepared free incoming states with the free outgoing states.

The free Hamiltonian can, in principle, be fixed arbitrarily. For the specific situation of the nuclear many-body problem, the free-particle Hamiltonian is defined by the translational term

$$H_0 = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} \ .$$

For other problems, like interaction between matter and radiation, interaction between fields, or even periodic interactions, the free Hamiltonian may appear in different forms [1,100,5,7,67,9].

For a wide range of physical systems, the interaction is represented by a short ranged potential $V(\mathbf{r})$. We shall make the following hypothesis on this potential:

(i) $V(\mathbf{r})$ is spherically symmetric.
(ii) $V(\mathbf{r}) \mapsto 0$, if $\mathbf{r} \mapsto \infty$.
(iii) $V(\mathbf{r})$ decreases sufficiently fast outside a domain.

More stringent conditions on the potential can be found, for instance, in the domain of nuclear physics [32,48], where $V(\mathbf{r})$ is also finite at the origin.

The treatment of this sort of potentials in ordinary quantum mechanics proceeds via a diagonalization in a basis of bound and scattering states. It means that any pure state can be written in terms of a linear combination of bound and scattering states. When resonances are present, any pure state can be written in terms of linear combinations of bound states, resonances and a background. Now, the question arises on how we can accommodate this decomposition from the mathematical point of view.

2.1. The Hilbert space of scattering states

Once we have introduced the dynamics which produces the scattering phenomena, we should define what it is commonly understood by scattering states. Let H be a self-adjoint Hamiltonian. This Hamiltonian may have bound states. These states are eigenvectors of H with real eigenvalues and they span a Hilbert space which is called $\mathscr{H}_{\rm d}$, where the subindex d stands for *discrete*. In addition to $\mathscr{H}_{\rm d}$, we have the Hilbert space $\mathscr{H}_{\rm c}$, which is the space of vectors orthogonal to all vectors in $\mathscr{H}_{\rm d}$. we say that $\mathscr{H}_{\rm c}$ is the continuous subspace of $\mathscr{H}$ with respect to H. Thus, the Hilbert space of pure states of the system decomposes as the direct orthogonal sum ($\oplus$) of $\mathscr{H}_{\rm d}$ and $\mathscr{H}_{\rm c}$:

$$\mathscr{H} = \mathscr{H}_{\rm d} \oplus \mathscr{H}_{\rm c} \ . \tag{1}$$

The continuous subspace of $\mathscr{H}$ with respect to H, which is $\mathscr{H}_{\rm c}$, has the following properties:

(i) There are enough vectors in $\mathscr{H}_{\rm c}$ in which H acts. The result of applying H to a vector in $\mathscr{H}_{\rm c}$ is also a vector in $\mathscr{H}_{\rm c}$. We then say that H *reduces*[1] $\mathscr{H}_{\rm c}$.

(ii) The operator H on $\mathscr{H}_{\rm c}$ is self-adjoint and its spectrum is purely continuous.

The space $\mathscr{H}_{\rm d}$ is spanned by the bound states of H. Consequently, the scattering states must be included in $\mathscr{H}_{\rm c}$. However, for some Hamiltonians, not all states of $\mathscr{H}_{\rm c}$, in a dense subspace of $\mathscr{H}_{\rm c}$, should be considered as scattering states. It can be realized when the continuous spectrum of H has a fractal section. In such a case, $\mathscr{H}_{\rm c}$ is decomposed into two mutually orthogonal parts, namely

$$\mathscr{H}_{\rm c} = \mathscr{H}_{\rm ac} \oplus \mathscr{H}_{\rm sc} \ ,$$

where the sub-indices ac and sc stand for absolutely continuous and singular continuous, respectively. The Hamiltonian H reduces both $\mathscr{H}_{\rm ac}$ and $\mathscr{H}_{\rm sc}$ and the fractal section of the spectrum of H is the spectrum of the restriction of H to $\mathscr{H}_{\rm sc}$. This spectrum is called the singular continuous spectrum of H. The spectrum of H in $\mathscr{H}_{\rm ac}$ is the *absolutely continuous* spectrum of H. These names are taken from the measure theory [98]. Therefore, we have the following decomposition:

$$\mathscr{H} = \mathscr{H}_{\rm d} \oplus \mathscr{H}_{\rm ac} \oplus \mathscr{H}_{\rm sc} \ . \tag{2}$$

Scattering states are not-bound regular vectors and therefore they belong to $\mathscr{H}_{\rm ac}$. In addition, they should have finite energy. This property holds, only, in a dense subspace of $\mathscr{H}_{\rm ac}$.[2]

---

[1] In mathematical terms, the restriction of the domain $\mathscr{D}_{\rm c}$ of H to $\mathscr{H}_{\rm c}$ is dense in $\mathscr{H}_{\rm c}$ and $H\psi \in \mathscr{H}_{\rm c}$, $\forall\, \psi \in \mathscr{D}_{\rm c}$.

[2] A subspace $\mathscr{D}$ of a Hilbert space $\mathscr{H}$ is dense if any vector in $\mathscr{H}$ can be arbitrarily approximated in norm by a vector in $\mathscr{D}$. Thus, for an arbitrary $\phi \in \mathscr{H}$ and for any $\epsilon > 0$, it exists $\psi \in \mathscr{D}$ such that

$$\|\phi - \psi\| < \epsilon.$$

In physical terms this inequality means that the state ϕ can be replaced by the state ψ within an accuracy ϵ.

2.2. The Møller wave operators

We assume that any scattering state is asymptotically free in the past. For any scattering state ϕ, it exists a free state ψ such that [3]

$$\lim_{t\to-\infty}\{e^{-itH}\phi - e^{-itH_0}\psi\} = 0\ . \tag{3}$$

As the limit in a Hilbert space is taken with respect to its norm, this is equivalent to say that

$$\lim_{t\to-\infty}\|e^{-itH}\phi - e^{-itH_0}\psi\| = 0\ . \tag{4}$$

Since the evolution operator is unitary, we have:

$$\lim_{t\to-\infty}\|\phi - e^{itH}e^{-itH_0}\psi\| = 0\ . \tag{5}$$

It is natural to assume that any state can be prepared as a free state, i.e. ψ is an arbitrary vector in $\mathscr{H}$. Then, we can define an operator which relates each scattering state ϕ with its corresponding asymptotically free state ψ as

$$\phi := \boldsymbol{\Omega}_{\mathrm{OUT}}\psi = \lim_{t\to-\infty} e^{itH}e^{-itH_0}\psi\ . \tag{6}$$

The existence of $\boldsymbol{\Omega}_{\mathrm{OUT}}$ depends on the potential $V(\mathbf{r})$, see [100].

Analogously, we also assume that any scattering state is asymptotically free in the future. This means that for any scattering state $\phi\in\mathscr{H}_{\mathrm{ac}}$, there exists a free state $\varphi\in\mathscr{H}$ such that

$$\lim_{t\to\infty}\{e^{-itH}\phi - e^{-itH_0}\varphi\} = 0\ . \tag{7}$$

If this happens for each $\varphi\in\mathscr{H}$, i.e., for each free state, then, there exists an operator $\boldsymbol{\Omega}_{\mathrm{IN}}$ defined as

$$\phi = \boldsymbol{\Omega}_{\mathrm{IN}}\varphi = \lim_{t\to\infty} e^{itH}e^{-itH_0}\varphi\ . \tag{8}$$

Again, the existence of $\boldsymbol{\Omega}_{\mathrm{IN}}$ depends on $V(\mathbf{r})$, see [100].

Along this paper we shall assume that no incoming scattering state can be trapped in the interaction region and become a bound state. Also, that no bound state can spontaneously decay and become a scattering state. This property is called *asymptotic completeness*. If asymptotic completeness applies, then the operators $\boldsymbol{\Omega}_{\mathrm{OUT}}$ and $\boldsymbol{\Omega}_{\mathrm{IN}}$ are unitary operators from $\mathscr{H}$ to $\mathscr{H}_{\mathrm{ac}}$ [1,100]. Therefore, their inverses exists from $\mathscr{H}_{\mathrm{ac}}$ to $\mathscr{H}$. As these operators are unitary their inverses coincide with their adjoints and

$$\boldsymbol{\Omega}_{\mathrm{IN}}^{-1} = \boldsymbol{\Omega}_{\mathrm{IN}}^{\dagger},\quad \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} = \boldsymbol{\Omega}_{\mathrm{OUT}}^{\dagger}\ .$$

The operators $\boldsymbol{\Omega}_{\mathrm{OUT}}$ and $\boldsymbol{\Omega}_{\mathrm{IN}}$ are the *Møller wave operators*. They have the following property: if

$$\boldsymbol{\Omega}\psi(0) = \varphi(0)\ ,$$

where $\boldsymbol{\Omega}$ can be either $\boldsymbol{\Omega}_{\mathrm{OUT}}$ or $\boldsymbol{\Omega}_{\mathrm{IN}}$, then,

$$\boldsymbol{\Omega}\psi(\tau) = \varphi(\tau)\ ,$$

[3] From hereon we take $\hbar = 1$, unless indicated explicitly.

where τ is an arbitrary time. The proof of this statement is very simple. Write

$$e^{itH}e^{-itH_0}\psi(\tau) = e^{itH}e^{-i(t+\tau)H_0}\psi(0) = e^{-i\tau H}e^{i(t+\tau)H}e^{-i(t+\tau)H_0}\psi(0) \ ,$$

taking limits as t goes to either $-\infty$ or ∞, we have

$$\boldsymbol{\Omega}\psi(\tau) = e^{-i\tau H}\varphi(0) = \varphi(\tau) \ .$$

This formula can also be written as

$$\boldsymbol{\Omega}e^{-i\tau H_0}\psi(0) = e^{-i\tau H}\varphi(0) = e^{-i\tau H}\boldsymbol{\Omega}\psi(0) \ .$$

If the free state $\psi(0)$ can be arbitrary chosen, we have that

$$\boldsymbol{\Omega}e^{-i\tau H_0} = e^{-i\tau H}\boldsymbol{\Omega} \ , \tag{9}$$

or in infinitesimal form [100],

$$\boldsymbol{\Omega}H_0 = H\boldsymbol{\Omega} \ . \tag{10}$$

In scattering theory we prepare a free state in the remote past. It evolves freely and at the time t this state would have been $\psi = \psi(t)$ if no interaction is present. As the particle in the state $\psi(t)$ enters in the interaction region, the state becomes $\boldsymbol{\Omega}_{\mathrm{IN}}\psi(t) = \phi(t)$. When the particle leaves the interaction region, its state is given by $\varphi(t)$ where $\boldsymbol{\Omega}_{\mathrm{OUT}}\psi(t) = \varphi(t)$. The relation between the free incoming state $\psi(t)$ and the outgoing state $\varphi(t)$ is given, at all times, by

$$\varphi(t) = \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}\boldsymbol{\Omega}_{\mathrm{IN}}\psi(t) \ . \tag{11}$$

If we assume asymptotic completeness, for any free incoming state ψ, there is a unique outgoing state φ.[4] As the Mller operators are unitary,[5] the operator

$$S := \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}\boldsymbol{\Omega}_{\mathrm{IN}} \tag{12}$$

is also unitary. Observe that it maps $\mathscr{H}$ onto itself. The operator S is the *S-operator*. Eq. (11) can also be written as

$$Se^{-itH_0}\psi(0) = e^{-itH_0}\varphi(0) = e^{-itH_0}S\psi(0) \ .$$

Since the choice of $\psi(0)$ is arbitrary, we have that

$$Se^{-itH_0} = e^{-itH_0}S \ , \tag{13}$$

or, in infinitesimal form,

$$SH_0 = H_0S \ \Leftrightarrow \ [S, H_0] = 0 \ . \tag{14}$$

Eq. (14) shows that the S-operator commutes with the free Hamiltonian H_0. This property has important consequences which will be commented upon in the next subsection.

---

[4] For the sake of convenience, we write $\psi \equiv \psi(0)$ and $\varphi \equiv \varphi(0)$.

[5] Strictly speaking, from the mathematical side, the Møller operators are unitary from the absolutely continuous subspace $H_{\mathrm{ac}}(H_0)$ onto $H_{\mathrm{ac}}(H)$, for H_0 and H, respectively.

2.3. The spectral theorem

Next, we are going to present, without proofs, important spectral properties of general self-adjoint operators. For practical purposes, we shall introduce here only those results of direct application to our formalism of resonance scattering.

Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$, a vector $\xi \in \mathscr{H}$ is said to be a cyclic vector of A if the sequence $\{\xi, A\xi, A^2\xi, \ldots, A^n\xi, \ldots\}$ forms a basis, in general not orthonormal, of $\mathscr{H}$. Self-adjoint operators having a cyclic vector, can be written as a multiplication operator, exactly like the position operator, on a certain Hilbert space. [6]

If A has absolutely continuous spectrum only, i.e., if the decomposition of $\mathscr{H}$ with respect to A given in (2) is $\mathscr{H} = \mathscr{H}_{\rm ac}$ with vanishing $\mathscr{H}_{\rm d}$ and $\mathscr{H}_{\rm sc}$, then, $\mathrm{d}\mu$ is the Lebesgue measure on $\sigma(A)$ [1,98].

Summarizing, we have a unitary mapping U:

$$U : \mathscr{H} \mapsto L^2[\sigma(A), \mathrm{d}\mu] \ , \tag{15}$$

such that if $\psi(x) \in L^2[\sigma(A), \mathrm{d}\mu]$ with $x\psi(x) \in L^2[\sigma(A), \mathrm{d}\mu]$, then

$$UAU^{-1}\psi(x) = x\psi(x) \ . \tag{16}$$

If S is an operator which commutes with A, then there exists a function $S(x)$ such that

$$USU^{-1}\psi(x) = S(x)\psi(x)$$

for any $\psi(x) \in L^2[\sigma(A), \mathrm{d}\mu]$ with $S(x)\psi(x) \in L^2[\sigma(A), \mathrm{d}\mu]$ [1].

Should A not have a cyclic vector on $\mathscr{H}$, then $\mathscr{H}$ can be decomposed as a direct sum of Hilbert spaces [84,98]

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots \oplus \mathscr{H}_n \ ,$$

such that A is an operator and it has a cyclic vector on each of the $\mathscr{H}_i$. Therefore, the spectral theorem applies on each of the $\mathscr{H}_i$. In this case, we can always find a unitary operator U, as in (15), and a function $f(x)$ such that

$$UAU^{-1}\psi(x) = f(x)\psi(x) \ .$$

This spectral theorem is one of the tools of our presentation which will allow us to define the Breit–Wigner energy distributions for the Gamow vectors.

2.4. A model for scattering

Let us now come back to the scattering problem: we shall assume that H_0 is the Hamiltonian of a free particle in three dimensions and that $V(\mathbf{r})$ is a potential which has the properties listed at the

[6] In fact, the spectral theorem [98] states that there exists a measure $\mathrm{d}\mu$ on the spectrum of A, $\sigma(A)$, and a unitary operator U from $\mathscr{H}$ into $L^2(\sigma(A), \mathrm{d}\mu)$, which is the space of square integrable functions from $\sigma(A)$ into $\mathbb{C}$, the set of complex numbers, with respect to the measure $\mathrm{d}\mu$, such that UAU^{-1} is the multiplication operator [1,98].

beginning of Section 2. Then,

(i) The Hamiltonians H_0 and $H = H_0 + V$ are self-adjoint.

(ii) As the potential is spherically symmetric, H commutes with the three components of the orbital angular momentum. In this case the Hilbert space $\mathscr{H}$, which in our case is $L^2[\mathbb{R}^3, \mathrm{d}^3\mathbf{r}]$, the space of all integrable complex functions on the three-dimensional real space $\mathbb{R}^3$, can be decomposed as

$$\mathscr{H} = \bigoplus_{l=0}^{\infty} \mathscr{H}_l \ , \tag{17}$$

where $\mathscr{H}_l$ is the Hilbert space of all the states with value l of the angular momentum. The operators H_0 and H are self-adjoint on each of the $\mathscr{H}_l$.

Let us take a fixed value of the orbital angular momentum l; thus $\mathscr{H}_l$ has a cyclic vector and therefore the spectral theorem applies on it. In particular, there exists a unitary mapping U from $\mathscr{H}_l$ to $L^2(\mathbb{R}^+, \mathrm{d}E)$ such that [7]

$$UH_0U^{-1}\psi_l(E) = E\psi_l(E), \quad \psi_l \in \mathscr{H}_l \ , \tag{18}$$

where ψ_l is the component of the vector ψ in the subspace $\mathscr{H}_l$.

The S-operator is a unitary operator on each of the $\mathscr{H}_l$. Since the S-operator commutes with H_0, there exists a function $S_l(E)$ such that

$$US_lU^{-1}\psi_l(E) = S_l(E)\psi_l(E) \ , \tag{19}$$

where S_l is the restriction of S to $\mathscr{H}_l$ and $\psi_l(E) \in L^2[\mathbb{R}^+, \mathrm{d}E]$. As U is unitary and S is also unitary on $\mathscr{H}_l$, we conclude that multiplication by $S_l(E)$ on $L^2[\mathbb{R}^+, \mathrm{d}E]$ is also a unitary application. This implies that [1]

$$S_l(E) = \mathrm{e}^{\mathrm{i}\delta_l(E)} \ , \tag{20}$$

where $\delta_l(E)$ is real for E real. An equation like (20) holds for all values of the angular momentum. The resulting functions $\delta_l(E)$ are the phase shifts.

Remark. From now on we shall work with a fixed value of the angular momentum and, for simplicity, we omit the label l unless otherwise stated. Note that the case $l = 0$ should be treated with care, since for it the centrifugal term vanishes in the radial form of both, the free and the total Hamiltonian.

(iii) On $\mathscr{H}_0$, both H_0 and H have spectrum $\sigma(H_0) = \sigma(H) = \mathbb{R}^+$.

(iv) The Møller wave operators exists and asymptotic completeness holds for the pair $\{H_0, H\}$ on $\mathscr{H}_0$. In particular, this means from (18) that

$$U\boldsymbol{\Omega}^{-1}H\boldsymbol{\Omega}U^{-1}\psi(E) = E\psi(E) \ , \tag{21}$$

where $\boldsymbol{\Omega}$ is either $\boldsymbol{\Omega}_{\mathrm{OUT}}$ or $\boldsymbol{\Omega}_{\mathrm{IN}}$. Thus, there are two unitary operators that fulfill the spectral theorem for H, which are

$$V_+ = U\boldsymbol{\Omega}_{\mathrm{OUT}}, \quad V_- = U\boldsymbol{\Omega}_{\mathrm{IN}} \ . \tag{22}$$

[7] $\mathbb{R}^+ \equiv [0, \infty)$. Therefore, $L^2(\mathbb{R}^+, \mathrm{d}E)$ is the space of complex square integrable functions on $\mathbb{R}^+$.

Therefore,

$$V_{\pm}HV_{\pm}^{-1}\psi(E) = E\psi(E)\ . \tag{23}$$

(v) The operator $S(E) := S_l(E)$ can be analytically continued as described in the next subsection.

2.5. The S-matrix formalism

We are now in the position to discuss the properties of the S-operator on $\mathscr{H}_l$. Although we shall always work in the energy representation, for the sake of clarity we shall discuss its properties on the p-plane, as a function of the variable $p = +\sqrt{2mE}$ [90,20]. We call this function $S(p)$ and it has the following properties:

1. As a consequence of causality [90], $S(p)$ admits an analytic continuation $S(z)$ on the complex plane. The possible singularities of $S(z)$ are poles that may be of three kinds [20,90]:

(i) Single poles in the positive imaginary axis of $S(z)$ that correspond to the bound states of H.

(ii) Single poles in the negative imaginary axis that correspond to virtual states.

(iii) Pairs of poles, in principle of any order, in the lower half-plane. Each of the poles of a pair has the same negative imaginary part and the same real part with opposite sign. Thus, if p_R is one of these two poles the other is $-p_R^*$ where the star denotes complex conjugation. These poles are called *resonance poles* and in general there is an infinite number of them [90].

2. This description has been made in the so-called p-plane or momentum plane. If we want to find an analogous description in the energy representation we have to make the change of variables given by $E = p^2/(2m)$. After this change of variables the p-plane is transformed into a Riemann surface. This mathematical construction is composed by two planes which have the origin in common. Both planes are called the first and the second Riemann sheet, respectively. The first and second sheet are connected by their positive semi-axis. The upper rim connects the upper half-plane of the first sheet with the lower half-plane of the second sheet, while the lower rim connects the lower half-plane in the first sheet with the upper half-plane in the second sheet.

Any function $S(z)$ on the p-plane can be transported to the Riemann surface by means of the relation $w = z^2/(2m)$. The function $S(w)$, defined on the Riemann surface, has the following properties:

(i) If p_R is a pole of $S(z)$ in the p-plane, $z_R = p_R^2/(2m)$ is a pole of $S(w)$ in the Riemann surface with the same multiplicity [5]. Thus, $S(w)$ has three types of singularities:

(a) simple poles in the negative semi-axis in the first sheet that correspond to the bound states of H,
(b) simple poles in the negative semi-axis of the second sheet that correspond to virtual states, and
(c) pairs of complex conjugate poles in the second sheet with positive real part.[8] They are the resonance poles. Each pair corresponds to a resonance in which the real part is the resonant energy and the modulus of the imaginary part is $\Gamma/2$ [20].

[8] The fact that the real part of the complex conjugate poles be positive is a consequence of causality. If we define resonances as poles of the extended resolvent [66,4] more general situations may arise. In particular, resonance poles may have negative real parts.

(ii) Let us call $S(E-\mathrm{i}0)$ the values of $S(w)$ on the lower rim of the cut. The boundary values of $S(w)$ on the upper rim of the cut are denoted by $S(E+\mathrm{i}0)$ and coincide with $S(E)$. These values are related:

$$S(E+\mathrm{i}0)=S^{*}(E-\mathrm{i}0)\ . \tag{24}$$

The above conditions (i) and (ii) are, in fact, the simplest which may be ascribed to the S matrix, in order to discuss the mathematical properties and the proper mathematical frame for Gamow vectors. The question about the existence of potentials which fulfill these conditions has been answered, positively, in Ref. [8]. Moreover, in Ref. [53] it is shown that it exists a class of realistic potentials which fulfill conditions on the corresponding S-matrix which are slightly different from conditions (i) and (ii) and that nevertheless yield to similar conclusions about Gamow vectors.

Remark. We have introduced the function $S(w)$, defined on a Riemann surface as it is usually done. However, what are relevant here are the analytic properties of $S(w)$ and not the geometric structure of the Riemann surface, which can be ignored. Then, we can consider that $S(w)$ is an analytic function on a complex plane without isolated singularities other than the bound poles on the negative semi-axis, with a discontinuity on the positive semi-axis given by (24). This function can be continued through the real axis into two different manners: from above to below and from below to above. These two analytic continuations are identical with poles at the locations of the virtual and resonance poles.

2.6. Rigged Hilbert spaces (RHS)

We shall define resonance states as eigenvectors of H with complex eigenvalues, located at the resonance poles. As self-adjoint operators in Hilbert spaces do not have complex eigenvalues, *resonant states cannot be vectors on a Hilbert space*. They belong to certain extensions of Hilbert spaces which are the rigged Hilbert spaces (RHS). We start with a definition of RHS.

A triplet of spaces [63,21,2,102,85,26,16,56]

$$\boldsymbol{\Phi}\subset\mathscr{H}\subset\boldsymbol{\Phi}^{\times} \tag{25}$$

is a rigged Hilbert space (RHS) if:

(i) The intermediate space $\mathscr{H}$ is an infinite-dimensional Hilbert space.[9]

(ii) The space $\boldsymbol{\Phi}$ is a topological vector space, which is dense in $\mathscr{H}$. This means that for any $\phi\in\mathscr{H}$ and for any positive number $\epsilon>0$, there is another vector $\varphi\in\boldsymbol{\Phi}$, such that $\|\phi-\varphi\|<\epsilon$. In other words, to any state ϕ in $\mathscr{H}$ it can be assigned a state φ in $\boldsymbol{\Phi}$, such that φ approaches ϕ with arbitrary accuracy. The space $\boldsymbol{\Phi}$ has its own topology, which is stronger than the topology that $\boldsymbol{\Phi}$ possesses as a subspace of $\mathscr{H}$. This can be explained by noticing that the topology in $\boldsymbol{\Phi}$ has more open sets, and, consequently, more neighborhoods and less convergent sequences. As a consequence,

[9] We shall always assume that our Hilbert spaces are separable.

the identity mapping

$$i : \boldsymbol{\Phi} \mapsto \mathscr{H} \ ,$$

such that $i(\varphi) = \varphi$ for all $\varphi \in \boldsymbol{\Phi}$, is continuous. The topology in $\boldsymbol{\Phi}$ is not given by a norm, but in the most simplest cases of physical interest, by a countable infinite family of norms. In this latter case, $\boldsymbol{\Phi}$ has the structure of a metric space.

(iii) The space $\boldsymbol{\Phi}^\times$ is the anti-dual space of $\boldsymbol{\Phi}$. The vectors of $\boldsymbol{\Phi}^\times$ are mappings, F, from $\boldsymbol{\Phi}$ into the set of complex numbers $\mathbb{C}$, i.e., $F : \boldsymbol{\Phi} \mapsto \mathbb{C}$, with the following properties:

(a) *Anti-linearity*: Given $F \in \boldsymbol{\Phi}^\times$, for $\varphi, \psi \in \boldsymbol{\Phi}$ and $\alpha, \beta \in \mathbb{C}$, we have that

$$F(\alpha\varphi + \beta\psi) = \alpha^* F(\varphi) + \beta^* F(\psi) \ ,$$

where the star denotes complex conjugation.

(b) *Continuity*: It implies, in particular, that if $\varphi_n \mapsto \varphi$ in $\boldsymbol{\Phi}$, then, $F(\varphi_n) \mapsto F(\varphi)$ in $\mathbb{C}$, if $F \in \boldsymbol{\Phi}^\times$.

The vectors in $\boldsymbol{\Phi}^\times$ are often called *functionals*. The action of $F \in \boldsymbol{\Phi}^\times$ on $\varphi \in \boldsymbol{\Phi}$ is usually expressed as $F(\varphi)$ or $\langle \varphi | F \rangle$. This second notation, which is the most usual among physicists, originates in Dirac notation [44]. Moreover, in a RHS, $\mathscr{H} \subset \boldsymbol{\Phi}^\times$, meaning means that any vector in $\mathscr{H}$ can be viewed as a functional on $\boldsymbol{\Phi}$. If $\psi \in \mathscr{H}$, then the functional F_ψ is defined as

$$\langle \varphi | F_\psi \rangle = \langle \varphi | \psi \rangle \ ,$$

where $\langle \varphi | \psi \rangle$ is the usual scalar product in $\mathscr{H}$. F_ψ is uniquely defined with ψ and it belongs to $\boldsymbol{\Phi}^\times$. RHS are useful, among other applications, for:

1. Giving a rigorous meaning to the Dirac formulation of quantum mechanics [44]. In this case, it is customary to demand that $\boldsymbol{\Phi}$ be nuclear. For a definition of nuclearity and the discussion of the nuclear spectral theorem, (see [60]). For the sake of completeness, we shall quote the Nuclear Spectral Theorem, according to which every observable, or set of commuting observables, has a complete set of generalized eigenvectors whose corresponding eigenvalues exhaust the whole spectrum of the observable. This is in agreement with the Dirac requirement [44,46,21,26,102,2,3,28,85].

2. Giving a proper mathematical meaning to the Gamow vectors, i.e., vector states which represent resonances [20,26,22,52].

3. Extending quantum mechanics to accommodate the *irreversible character* of certain quantum processes such as decay processes [95,97,96,13].

4. Providing an appropriate context for the spectral decompositions of the Frobenius Perron operator for certain chaotic systems in terms of Pollicot–Ruelle resonances [14].

5. Extending the formalism of statistical mechanics in order to include generalized states and singular structures on it. This can be achieved with the use of the rigged Liouville spaces (RLS) [11,12].

6. Defining some elements that appear in the axiomatic theory of quantum fields: Wightman functional, Borchers algebra, generalized states, etc. [19,54,7].

7. Dealing with physical problems requiring the use of distributions. In fact, distributions are well known to be objects in the dual of a nuclear locally convex space [60–62,84].

Rigged Hilbert spaces have the following properties:

Property 1. Let A be an operator on $\mathscr{H}$, $A^\dagger$ be its adjoint [10] and $\mathscr{D}(A^\dagger)$ be the domain of $A^\dagger$. Then,
(i) The domain, $\mathscr{D}(A^\dagger)$, contains the space $\boldsymbol{\Phi}$, i.e., $\boldsymbol{\Phi} \subset \mathscr{D}(A^\dagger)$.
(ii) For each $\varphi \in \boldsymbol{\Phi}$, we have that $A^\dagger \varphi \in \boldsymbol{\Phi}$. We say that $A^\dagger \boldsymbol{\Phi} \subset \boldsymbol{\Phi}$.
(iii) The operator $A^\dagger$ is continuous on $\boldsymbol{\Phi}$ with the own topology on $\boldsymbol{\Phi}$.
Then, the operator A can be extended [11] into the antidual $\boldsymbol{\Phi}^\times$ by the duality formula:

$$\langle A^\dagger \varphi | F \rangle = \langle \varphi | AF \rangle, \quad \forall \varphi \in \boldsymbol{\Phi}, \ \forall F \in \boldsymbol{\Phi}^\times \ . \tag{26}$$

Moreover, A is linear and continuous on $\boldsymbol{\Phi}^\times$.

This property also applies when A is self-adjoint. In this case $A^\dagger = A$ and we do not need to write $A^\dagger$ in (26). Note that this property will allow us to construct continuous extensions of operators which are not continuous on $\mathscr{H}$. When A is self-adjoint it is always possible to find a RHS with this property [84].

Property 2. Let A be an operator with the properties described in Property 1. A complex number λ is a *generalized eigenvalue of A* if for any $\varphi \in \boldsymbol{\Phi}$ and for some non-zero $F \in \boldsymbol{\Phi}^\times$, we have that

$$\langle A^\dagger \varphi | F \rangle = \lambda \langle \varphi | F \rangle \ . \tag{27}$$

After Eqs. (26) and (27) we have that if λ is a generalized eigenvalue of A, then,

$$\langle \varphi | AF \rangle = \lambda \langle \varphi | F \rangle$$

for all $\varphi \in \boldsymbol{\Phi}$. If we omit this arbitrary φ, we have that

$$AF = \lambda F \ .$$

Therefore, a generalized eigenvalue of A is just an eigenvalue of the extension of A into $\boldsymbol{\Phi}^\times$. As this extension is a linear mapping, μF is also an eigenvector of A on $\boldsymbol{\Phi}^\times$ with eigenvalue λ, for any complex number $\mu \neq 0$. The functional F is often called a generalized eigenvector or a generalized eigenfunctional of A with eigenvalue λ.

Property 3. A result due to Gelfand and Maurin [62,84] states the following: Let A be a self-adjoint operator on $\mathscr{H}$ with continuous spectrum $\sigma(A)$. Although it is not necessary, we may assume that the spectrum is purely continuous. Then, there exists a RHS, $\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^\times$, such that
(i) $A\boldsymbol{\Phi} \subset \boldsymbol{\Phi}$, and A is continuous on $\boldsymbol{\Phi}$. Therefore, A can be extended by the duality formula (26) to the anti-dual space $\boldsymbol{\Phi}^\times$.
(ii) There exists a measure $\mathrm{d}\mu$ on $\sigma(A)$, which can be chosen to be the Lebesgue measure if the spectrum is absolutely continuous, such that for almost all $\lambda \in \sigma(A)$ with respect to $\mathrm{d}\mu$, there exists a nonzero functional $F_\lambda \in \boldsymbol{\Phi}^\times$ such that

$$AF_\lambda = \lambda F_\lambda \ . \tag{28}$$

[10] We always assume that A has a dense domain on $\mathscr{H}$.
[11] The extension of the operator A is often denoted as $A^\times$ [20,22]. Nevertheless, we shall use the same notation for A and for its extension to $\boldsymbol{\Phi}^\times$ in order to keep the notation as simple as possible. Note that this is a true extension of A in the sense that the restriction of the extended operator to the domain of A gives A again.

This means that the points in the continuous spectrum of A are eigenvectors of the extension of A into $\boldsymbol{\Phi}^{\times}$. *However, these eigenvectors do not belong to the Hilbert space* $\mathscr{H}$. In addition, the generalized eigenvectors of A with eigenvalues covering the continuous spectrum of A form a complete set. For all $\varphi, \phi \in \boldsymbol{\Phi}$, we have that

$$\langle \varphi | A\phi \rangle = \int_{\sigma(A)} \lambda F_\lambda(\varphi)[F_\lambda(\phi)]^* \, \mathrm{d}\mu \tag{29}$$

and that

$$\langle \varphi | \phi \rangle = \int_{\sigma(A)} F_\lambda(\varphi)[F_\lambda(\phi)]^* \, \mathrm{d}\mu \ . \tag{30}$$

Eqs. (29) and (30) are spectral representations, so that we can define functions $f(A)$ of A by using the formula

$$\langle \varphi | f(A)\phi \rangle = \int_{\sigma(A)} f(\lambda) F_\lambda(\varphi)[F_\lambda(\phi)]^* \, \mathrm{d}\mu \ , \tag{31}$$

provided this integral converges. If the spectrum of A has no singular part, then, $\mathrm{d}\mu$ can be chosen to be the Lebesgue measure. Then, following the conventional Dirac notation, we can write $F_\lambda = |\lambda\rangle$ and Eq. (31) can be written as

$$\langle \varphi | f(A)\phi \rangle = \int_{\sigma(A)} f(\lambda) \langle \varphi | \lambda \rangle \langle \lambda | \phi \rangle \, \mathrm{d}\lambda \ , \tag{32}$$

where we have used the convention $\langle \lambda | \phi \rangle = \langle \phi | \lambda \rangle^*$. This equation is often written as

$$f(A) = \int_{\sigma(A)} f(\lambda) |\lambda\rangle \langle \lambda| \, \mathrm{d}\lambda \ , \tag{33}$$

Eqs. (29) and (30) are particular cases of (32).

2.7. Examples of RHS

Example 1. Let us consider the set of functions S mapping the real line $\mathbb{R}$ into the complex plane $\mathbb{C}$ satisfying the following properties [98,61]:

(i) If $f(x) \in S$, $f(x)$ admits derivatives of all orders at all points. This implies that all derivatives of $f(x)$, $f^{(n}(x)$, $n = 0, 1, 2, \ldots$ exists and are continuous functions.

(ii) Each function $f(x) \in S$ and its derivatives at all orders go to zero at the infinity faster than the inverse of any polynomial. In mathematical terms, this property can be written as

$$\lim_{x \to \pm\infty} \left| x^n \frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x) \right| = 0, \quad n, m = 0, 1, 2, \ldots \ .$$

From this definition, we see that if $f(x)$ belongs to S, all its derivatives also are in S. The space S is a vector space, since the sum of functions in S and their products by complex numbers are in S. It is customary to endow S with a topology such that a sequence $f_n(x) \in S$ converges to

a function $f(x) \in S$ if and only if

$$\left|\left|\left(x^2 + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^p [f_n(x) - f(x)]\right|\right| \mapsto 0, \quad p = 0, 1, 2, \ldots , \tag{34}$$

where the norm here is given by

$$\|f(x)\| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x} \ .$$

This norm is well defined for all functions $f(x) \in S$.

A typical example of a function in S is $f(x) = \exp\{-x^2\}$. The space S is called the Schwartz space and the functions on S are the Schwartz functions. The space S has the following properties:

(a) If $f(x) \in S$, then $f(x)$ is square integrable with respect to the Lebesgue measure on $\mathbb{R}$. The functions with this latter property form a Hilbert space called $L^2(\mathbb{R})$. Thus, $S \subset L^2(\mathbb{R})$.

(b) The space S is dense in $L^2(\mathbb{R})$. Therefore, each function in $L^2(\mathbb{R})$ can be approached by a function in S with arbitrary accuracy.

(c) If $f_n(x) \mapsto f(x)$ in S, then $\|f_n(x) - f(x)\| \mapsto 0$. This is the case $p = 0$ in Eq. (34). Therefore all convergent sequences in S also converge in $L^2(\mathbb{R})$, but the converse is not true in general. S has less convergent sequences than $L^2(\mathbb{R})$. The identity map from S into $L^2(\mathbb{R})$ is continuous.

(d) Let us consider the set $S^\times$ of continuous anti-linear functions from S into $\mathbb{C}$. Then, $S^\times$ is a vector space that contains $L^2(\mathbb{R})$ and that can be endowed with its own topology in such a way that the identity mapping from $L^2(\mathbb{R})$ into $S^\times$ ($i(\varphi) = \varphi$, $\forall \varphi \in L^2(\mathbb{R})$) is continuous. In fact, the topology that $L^2(\mathbb{R})$ inherits from $S^\times$ is weaker (it has less neighborhoods) than the topology of the Hilbert space. Therefore, there are more convergent sequences in $S^\times$ than in $L^2(\mathbb{R})$.

From all these properties we conclude that [26]

$$S \subset L^2(\mathbb{R}) \subset S^\times \tag{35}$$

is a RHS. Typical functionals in $S^\times$ are the Dirac delta:

$$F[f(x)] = \int_{-\infty}^{\infty} f^*(x)\delta(x - x_0) \, \mathrm{d}x = f^*(x_0), \quad \forall f(x) \in S$$

and the Fourier transform at the point $k \in \mathbb{R}$:

$$F[f(x)] = \int_{-\infty}^{\infty} f^*(x)\mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x = \hat{f}(k), \quad \forall f(x) \in S \ .$$

Observe that F is anti-linear in both cases.

Example 2. Let $f(z)$ be a complex analytic function on the open upper half-plane $\mathbb{C}^+$

$$\mathbb{C}^+ = \{z \in \mathbb{C};\ z = x + \mathrm{i}y;\ y > 0\} \ .$$

We say that $f(z)$ is a Hardy function on $\mathbb{C}^+$ if and only if [45,65,69,70,92,110]:

(i) For each $y > 0$ the function of real variable x, $f(x + \mathrm{i}y)$, is square integrable, i.e.,

$$I^+(y) = \int_{-\infty}^{\infty} |f(x + \mathrm{i}y)|^2 \, \mathrm{d}x < \infty \ . \tag{36}$$

(ii) Integrals (36) are uniformly bounded by the same upper bound K:

$$\sup_{y>0} I^+(y) = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+\mathrm{i}y)|^2 \,\mathrm{d}x < K < \infty \ . \tag{37}$$

Hardy functions on the upper half-plane $\mathbb{C}^+$ have the following properties:

(a) they form a vector space, $\mathscr{H}_+^2$. Functions in $\mathscr{H}_+^2$ have boundary values on the real axis. For each $f(z) \in \mathscr{H}_+^2$, the set of these boundary values form a function $f(x)$. This function is defined taking the limit

$$\lim_{z \to x} f(z) = f(x) \ .$$

The function $f(x)$ is square integrable and

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x \leqslant K < \infty \ , \tag{38}$$

where the constant K is the same in (38) and in (37). The function $f(z)$ determines uniquely the function $f(x)$.

(c) If $f(x)$ is the function of the boundary values of a Hardy function $f(z)$ on $\mathbb{C}^+$, we can recover all the values of $f(z)$ at the points $z \in \mathbb{C}^+$ by means of the following equation:

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{f(x)\,\mathrm{d}x}{x - z} \ . \tag{39}$$

This result can be obtained by using the Cauchy theorem [45,65,69,70] and it is called the Titchmarsh theorem. We see that a Hardy function determines uniquely the function of its boundary values and vice versa. Consequently, we can identify both $f(z)$ and $f(x)$, and we shall do it in the sequel.

(c) We need a criterium according to which we can determine whether a function $f(x) \in L^2(\mathbb{R})$ is a Hardy function. This criterium is provided by the Paley–Wienner [92,45,65,69,70] theorem which states that a function $f(x) \in L^2(\mathbb{R})$ is in $\mathscr{H}_+^2$ if and only if its inverse Fourier transform vanishes on the positive semi-axis. Moreover, if $\mathbb{R}^- = (-\infty, 0]$ and $L^2(\mathbb{R}^-)$ is the space of square integrable functions on $\mathbb{R}^-$, the Paley–Wienner theorem establishes that the Fourier transform is unitary from $L^2(\mathbb{R}^-)$ onto $\mathscr{H}_+^2$.

(d) Let S^- be the space of all functions in S which vanish on the positive semi-axis $\mathbb{R}^+ = [0, \infty)$. Take the space of the Fourier transform of the functions in S^-. Due to the Paley–Wienner theorem and the fact that the Fourier transform of a Schwartz function is also a Schwartz function, we conclude that the Fourier transform of the space S^- is the space $S \cap \mathscr{H}_+^2$. As $S \cap \mathscr{H}_+^2$ is a subspace of S, we can use in $S \cap \mathscr{H}_+^2$ the topology of S.

(e) The space $\mathscr{H}_+^2$ is a subspace of $L^2(\mathbb{R})$. Furthermore, $\mathscr{H}_+^2$ is a Hilbert space when we give to functions in $\mathscr{H}_+^2$ the $L^2(\mathbb{R})$ norm [45,65,69,70].

From all these properties and some mathematical machinery, taking some other properties into account, as the fact that S^- is dense in $L^2(\mathbb{R}^-)$, we conclude that [26,52]

$$S \cap \mathscr{H}_+^2 \subset \mathscr{H}_+^2 \subset (S \cap \mathscr{H}_+^2)^\times \ , \tag{40}$$

is a RHS. Eq. (40) implies that: $S \cap \mathscr{H}_+^2$ is a realization of $\boldsymbol{\Phi}$, $\mathscr{H}_+^2$ is a realization of the Hilbert space $\mathscr{H}$ and $(S \cap \mathscr{H}_+^2)^\times$ is a realization of the anti-dual space $\boldsymbol{\Phi}^\times$.

Example 3. Let us take now the space of all complex analytic functions $f(z)$ on the lower half-plane $\mathbb{C}^-$

$$\mathbb{C}^- = \{z \in \mathbb{C};\ z = x - \mathrm{i}y;\ y > 0\}\ ,$$

such that

(i) For each $y > 0$, the function of real variable x, $f(x - \mathrm{i}y)$, is square integrable, i.e.,

$$I^-(y) = \int_{-\infty}^{\infty} |f(x - \mathrm{i}y)|^2\, \mathrm{d}x < \infty\ . \tag{41}$$

(ii) Integrals (41) are uniformly bounded by the upper bound K:

$$\sup_{y>0} I^-(y) = \sup_{y>0} \int_{-\infty}^{\infty} |f(x - \mathrm{i}y)|^2\, \mathrm{d}x < K' < \infty\ . \tag{42}$$

These functions are called the Hardy functions on the lower half-plane. They form a vector space which is denoted as $\mathscr{H}^2_-$, which has very similar properties as $\mathscr{H}^2_+$. In particular:

(a) The boundary values on the real axis of every function $f(z) \in \mathscr{H}^2_-$ determines a square integrable function $f(x)$ such that

$$\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x < K' < \infty\ .$$

(b) We can recover the values of $f(z)$ for each $z \in \mathbb{C}^-$ by using the Titchmarsh formula:

$$f(z) = -\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{f(x)\, \mathrm{d}x}{x - z}\ . \tag{43}$$

Note on the minus sign in (43) which does not appears in (39). Thus, we can identify each function $f(z) \in \mathscr{H}^2_-$ with the function given by its boundary values $f(x)$.

(c) A version of the Paley–Wienner theorem also applies here. The space $\mathscr{H}^2_-$ is the space of the Fourier transform of the functions in $L^2(\mathbb{R}^+)$. If S^+ is the subspace of the Schwartz space of all functions that vanish on the negative semi-axis $\mathbb{R}^- = (0, \infty]$, the Fourier transform of S^+ is given by $S \cap \mathscr{H}^2_-$.

(d) The space $\mathscr{H}^2_-$ is a subspace of $L^2(\mathbb{R})$ and is a Hilbert space with the $L^2(\mathbb{R})$ norm.

(v) It holds that $L^2(\mathbb{R}) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^-)$. Using the properties of the Fourier transform we conclude that

$$L^2(\mathbb{R}) = \mathscr{H}^2_+ \oplus \mathscr{H}^2_-\ ,$$

where $\oplus$ denotes orthogonal sum.

(f) If we endow $S \cap \mathscr{H}^2_-$ with the topology of S, we can show that

$$S \cap \mathscr{H}^2_- \subset \mathscr{H}^2_- \subset (S \cap \mathscr{H}^2_-)^\times\ , \tag{44}$$

is a new RHS. Observe that, again, $S \cap \mathscr{H}^2_-$ is a realization of $\boldsymbol{\Phi}$, $\mathscr{H}^2_-$ is a realization of the Hilbert space $\mathscr{H}$ and, finally, $(S \cap \mathscr{H}^2_-)^\times$ is a realization of the anti-dual space $\boldsymbol{\Phi}^\times$.

(g) Let $f(z)$ be a function either in $\mathscr{H}^2_+$ or in $\mathscr{H}^2_-$. For large values of $|z|$ we have [70]

$$|f(z)| \approx |z|^{-1/2}\ . \tag{45}$$

This gives the asymptotic behavior of a Hardy function on a half-plane.

(h) A theorem due to van Winter [112] shows that any function $f(z)$ in $\mathscr{H}^2_+$ or in $\mathscr{H}^2_-$ is uniquely determined by its boundary values on the positive semi-axis $\mathbb{R}^+ = [0, \infty]$. The values on the open half-plane and the negative real axis can be recovered from its values on $\mathbb{R}^+$. Moreover, if we called

$$S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+} \ , \tag{46}$$

to the spaces of functions that are restrictions to $\mathbb{R}^+$ of functions in $S \cap \mathscr{H}^2_\pm$, these spaces are dense in $L^2(\mathbb{R}^+)$ although $S \cap \mathscr{H}^2_\pm$ are not dense in $L^2(\mathbb{R}^+)$ [112,26]. This is a rather surprising result which will allow us to construct the next example (see Example 4). Note that the functions in $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ are just the boundary values of the functions in $S \cap \mathscr{H}^2_\pm$, where we are ignoring the values of these functions on $\mathbb{R}^- = (-\infty, 0]$.

(j) If $\psi_\pm(E) \in \mathscr{H}^2_\pm$, then, $[\psi_\pm(E)]^* \in \mathscr{H}^2_\mp$, where the star denotes complex conjugation. This means that the complex conjugate of a Hardy function on a half-plane is a Hardy function on the other.

Example 4. After the above comments, we can construct two mappings, namely,

$$\theta_\pm : S \cap \mathscr{H}^2_\pm \mapsto S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+} \ , \tag{47}$$

where $\theta_\pm$ carries a function $f(x) \in S \cap \mathscr{H}^2_\pm$ to the function of its boundary values on $\mathbb{R}^+$. As these boundary values determine the whole function, as a consequence of the van Winter theorem [112], the mappings $\theta_\pm$ have an inverse $\theta_\pm^{-1}$. This is non-trivial and it has interesting consequences, as we shall show later.

Take now spaces (46). A sequence $f_n(x) \in S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ converges to a function $f(x) \in S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ if and only if [26]

$$\theta_\pm^{-1}[f_n(x)] \mapsto \theta_\pm^{-1}[f(x)] \ ,$$

in $S \cap \mathscr{H}^2_\pm$. We say that $\theta_\pm$ transport the Schwartz topology from $S \cap \mathscr{H}^2_\pm$ to $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$. With this definition the mappings $\theta_\pm$ as well as the mappings $\theta_\pm^{-1}$ are continuous. We conclude that [26,52]

$$S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset (S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+})^\times \tag{48}$$

are two new RHS.

It is important to observe that the mappings $\theta_\pm$ are not unitary because do not preserve scalar products. If $f(x) \in S \cap \mathscr{H}^2_\pm$, then $f(x) \in L^2(\mathbb{R})$, and $\theta_\pm[f(x)] \in L^2(\mathbb{R}^+)$. Thus,

$$\|f(x)\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x$$

and

$$\|\theta_\pm[f(x)]\|^2 = \int_0^{\infty} |f(x)|^2 \, \mathrm{d}x < \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \|f(x)\|^2 \ .$$

This inequality is *strict*, i.e., the identity does not hold, because a Hardy function cannot vanish on any interval. [12]

We can now define one to one onto mappings:

$$\theta_\pm^\times : (S \cap \mathscr{H}^2_\pm)^\times \mapsto (S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+})^\times \ , \tag{49}$$

[12] Otherwise it would be the zero function.

by using the duality formula

$$\langle \theta_\pm \varphi_\pm | F_\pm \rangle = \langle \varphi_\pm | \theta_\pm^\times F_\pm \rangle, \quad \forall \varphi_\pm \in S \cap \mathscr{H}_\pm^2, \ \forall F_\pm \in (S \cap \mathscr{H}_\pm^2)^\times \ . \tag{50}$$

The mappings $\theta_\pm^\times$ and their respective inverses $\theta_{\pm\times}^{-1}$ are continuous [26]. Due to the non-unitarity of $\theta_\pm$, the mappings $\theta_\pm^\times$ are not extensions of the mappings $\theta_\pm$.

The RHS (48) will be the proper spaces to define the Gamow vectors. We shall see this in the next section.

2.8. Summary

The mathematical concepts presented in this section are related to the structure of RHS and the construction of RHS for relevant examples. The transformations $\theta_\pm$ needed to define the basic spaces Φ, or space of test vectors, and $\Phi^\times$, its dual, for certain cases of physical interest, have been presented. A brief list of the relevant objects is the following:

$\mathscr{H} = \oplus_n \mathscr{H}_n$ (Hilbert space) ,

$\boldsymbol{\Omega}_{\rm OUT}$, $\boldsymbol{\Omega}_{\rm IN}$ (Møller wave operators) ,

$S = \boldsymbol{\Omega}_{\rm IN} \boldsymbol{\Omega}_{\rm OUT}^\dagger$ (S-operator) ,

$\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^\times$ (rigged Hilbert spaces) ,

$\mathscr{H}_\pm^2$ (Hardy spaces) ,

$S \cap \mathscr{H}_\pm^2$ (Hardy–Schwartz Functions) ,

$\theta_\pm, \theta_\pm^\times$ (Mappings) ,

$S \cap \mathscr{H}_\pm^2|_{\mathbb{R}^+}$ (Restrictions to $\mathbb{R}^+$ of functions in $S \cap \mathscr{H}_\pm^2$) .

3. Physical concepts

The first known example of a quantum decay process is natural radioactivity, a process where the atomic nucleus decays by the emission of particles. Under the hypothesis that the decay probability on a radioactive sample does not depend on the past history of the sample, one concludes that if at the time $t = 0$ there exist a population of $N(0)$ nuclei, the number of remaining nuclei at a time $t > 0$ is given by

$$N(t) = N(0) \mathrm{e}^{-\gamma t} \ , \tag{51}$$

where $\gamma = \tau^{-1}$ is the inverse of the half-life τ of the considered radioactive nucleus.

From the point of view of quantum mechanics, the relevant quantity is the *non-decay probability*. Firstly, we define the *non-decay amplitude*, which is the scalar product of the decaying state,[13] $\psi = \psi(0)$ at $t = 0$, times the decaying state, $\psi(t)$, at $t > 0$. Thus, the non-decay amplitude is [50]

$$A(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} | \psi \rangle = \langle \psi(0) | \psi(t) \rangle \ , \tag{52}$$

and the non-decay probability is

$$P(t) = |A(t)|^2 = |\langle \psi | \mathrm{e}^{-\mathrm{i}tH} | \psi \rangle|^2 \ . \tag{53}$$

It means that if we have a radioactive sample with $N(0)$ nuclei at $t = 0$, the number of mother nuclei which still remain after a time $t > 0$ is given by

$$N(t) = N(0)P(t) \ .$$

The question on whether $P(t)$ is exponentially decaying at all values of time, as suggested by experiments on radioactive nuclei, can be answered by noticing that:

(i) For $t = 0$, the derivative of $P(t)$ with respect to t, $P'(0)$, is zero [50]. This implies that for small values of t:

$$P(t) > \mathrm{e}^{-\gamma t} \ .$$

As a direct consequence of this behavior of $P(t)$, one has the so-called Zeno effect: If a decaying particle is continuously checked to see whether it has decayed or not, it never decays [109,41]. This is somehow logical, since a system which is constantly measured will not evolve. However, the formal derivation of Zeno effect is not so simple and it requires the use of functional analysis [109,41].

(ii) For intermediate times, starting from a certain time t_1 smaller than the half-life, to a time t_2 larger than the half-life, the non-decay probability is exponential: $P(t) \approx \mathrm{e}^{-\gamma t}$. The time t_1 needed to begin with the exponential behavior is often called the Zeno time.

(iii) For very large times, $P(t) \approx At^{-n}$, where A is a positive constant and n is a positive number. This fact has been discovered by Khalfin [68] and it is thereafter called the Khalfin effect.

Comments. Deviations of $P(t)$ from the exponential decaying law for very small and very large times are obtained by using two hypothesis: the Hamiltonian which gives the dynamics of the system is semibound and the energy has a finite mean value $\langle \psi | H | \psi \rangle$ on the decaying state[14] ψ.

It has been shown that the Khalfin effect results from the fact that decay products recombine to form the original decaying state [50]. Repeated measurements on the system and the interaction with the environment may prevent the occurrence of this effect [50].

We can obtain a purely exponentially decaying state, if and only if its energy distribution is a Lorentzian or Breit–Wigner distribution of the type [50,35,49,47,20]

$$|\psi(E)|^2 = \frac{\gamma}{\pi} \frac{1}{(E - E_{\mathrm{R}})^2 + (2\gamma)^2} \ , \tag{54}$$

where E_{R} is the resonant energy or the energy difference between the undecayed system and the decay products. Observe that the Breit–Wigner distribution is different from zero on the whole real

[13] We assume for simplicity that the decaying state is a pure state.

[14] This is equivalent to say that ψ is in the domain of H.

axis, $-\infty < E < \infty$. This implies that, for a state that decays exponentially at all values of time, the Hamiltonian H cannot be semi-bound and its spectrum must cover the whole real axis [47]. In consequence, if exponentially decaying states are truly physical states, then, either the Hamiltonian has a spectrum which covers the *whole* real axis or the decaying state *is not a vector in the Hilbert space* on which H acts. If the first situation realizes, we are forced to discard Hamiltonians of the kind $H = \mathbf{p}^2/(2m) + V(\mathbf{x})$, which are nonetheless those describing resonance scattering. The latter situation implies the need to enlarge the Hilbert space. This is precisely what should be done in considering RHS.

3.1. Non-relativistic Gamow vectors

Here, we return to the model of resonance scattering introduced in Sections 2.4 and 2.5. The model has, in general, an infinite number of resonances, and this makes it rather complicated for a first definition of the Gamow vectors and its properties. Instead, we shall introduce a toy model [26]. We shall assume the existence of a resonance scattering which obey the following conditions:

(i) The resonance scattering is produced by a perturbation V on the free Hamiltonian H_0; therefore, we have two dynamics given by H_0 and $H = H_0 + V$, respectively.

(ii) The spectrum of both H_0 and H is simple, absolutely continuous and is $\mathbb{R}^+ = [0, \infty)$. [15]

(iii) The Møller wave operators exists and asymptotic completeness holds.

(iv) The S operator in the energy representation is a function of E, $S(E)$, whose analytic continuation has a pair of simple poles in the second sheet of the Riemann surface introduced in Section 2.5. These poles are located at the points

$$z_R = E_R - i\Gamma \quad \text{and} \quad z_R^* = E_R + i\Gamma \ . \tag{55}$$

(v) On the second sheet, the growing of $S(E)$ at infinity is not faster than exponential. In particular, this happens if for $|E|$ sufficiently large, it exists a polynomial $P(E)$ such that,

$$|S(E)| \leqslant |P(E)| \ . \tag{56}$$

Once we have introduced the model, let us show how to define the Gamow vectors and their properties. According to our definition, Gamow vectors are the state vectors for exponentially decaying resonance states. [16] We have already mentioned that Gamow vectors should live in suitable extensions of a Hilbert space given by the antidual in a RHS. Therefore, our first task is to define this RHS.

Since the spectrum of H_0 is simple and absolutely continuous, the spectral theorem [98] determines that there exists a unitary operator U acting from the Hilbert space $\mathscr{H}$, in which H_0 applies, [17] onto $L^2(\mathbb{R}^+)$ the space of square integrable functions on the spectrum of H_0 that we assume is $\mathbb{R}^+$. The image by U of H_0 is the multiplication operator on $L^2(\mathbb{R}^+)$, see (15) and (16). In our case,

[15] Both, H_0 and H may have bound states. In this case, we simply have to replace in the sequence the Hilbert space $\mathscr{H}$ by its absolutely continuous subspace with respect the operator which has bound states.

[16] We shall see later that those are not the only examples of Gamow vectors. New types of Gamow vectors arise when the resonance poles are multiple.

[17] Or its absolutely continuous part with respect to H_0 if H_0 has eigenvalues. This happens for instance in the Friedrichs model we shall discuss later.

we can write

$$U : \mathscr{H} \mapsto L^2(\mathbb{R}^+) \ .$$

For all $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, the function $\psi(E) = U\psi$ is the wave function of the state vector ψ *in the energy representation*. Also, if ψ is in the domain of H_0, we have that

$$[UH_0U^{-1}]\psi(E) = E\psi(E) \ ,$$

i.e., the image by U of H_0 in $L^2(\mathbb{R}^+)$ is the multiplication operator. Since U *diagonalizes* the operator H_0, we design the diagonal form by $\mathscr{E}$, such that $\mathscr{E}\psi(E) = E\psi(E)$. The operator $\mathscr{E}$ is the multiplication operator on $L^2(\mathbb{R}^+)$ and $\mathscr{E} = UH_0U^{-1}$.

Take now $V_\pm$ as in (22). Obviously $V_\pm : \mathscr{H} \mapsto L^2(\mathbb{R}^+)$ and $V_\pm$ are unitary operators. After (23) we have that $V_\pm H V_\pm^{-1} = \mathscr{E}$. Therefore, both operators $V_\pm$ diagonalize H.

Let us define

$$\boldsymbol{\Phi}_\pm = V_\pm^{-1}[S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}] \ . \tag{57}$$

Now, we transport the topology from $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ into $\boldsymbol{\Phi}_\pm$. This means that the sequence $(\psi_n)_\pm$ converges to $\psi_\pm$ ($(\psi_n)_\pm \mapsto \psi_\pm$) in $\boldsymbol{\Phi}_\pm$ if and only if the sequence $V_\pm(\psi_n)_\pm$ converges to $V_\pm\psi_\pm$ ($V_\pm(\psi_n)_\pm \mapsto V_\pm\psi_\pm$) in $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$.

Once we have endowed the spaces $\boldsymbol{\Phi}_\pm$ with a topology, we automatically find their corresponding anti-duals $\boldsymbol{\Phi}_\pm^\times$. After the van Winter theorem [112], we know that $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$. Since $V_\pm$ are unitary, $\boldsymbol{\Phi}_\pm$ are both dense in $\mathscr{H}$.[18] Then, we have two new RHS:

$$\boldsymbol{\Phi}_\pm \subset \mathscr{H} \subset \boldsymbol{\Phi}_\pm^\times \ . \tag{58}$$

The properties of $\mathscr{E}$ on $L^2(\mathbb{R}^+)$ [26] imply the following:

(i) The operator H does not lead out of $\boldsymbol{\Phi}_\pm$, i.e., $H\boldsymbol{\Phi}_\pm \subset \boldsymbol{\Phi}_\pm$.

(ii) The operator H is continuous on $\boldsymbol{\Phi}_\pm$. This implies that we can extend H into a continuous operator on $\boldsymbol{\Phi}_\pm^\times$. This can be done by using the duality formula (26):

$$\langle H\psi_\pm | F_\pm\rangle = \langle \psi_\pm | HF_\pm\rangle, \quad \forall\psi_\pm \in \boldsymbol{\Phi}_\pm; \ \forall F_\pm \in \boldsymbol{\Phi}_\pm^\times \ . \tag{59}$$

The extension of the operator H to $\boldsymbol{\Phi}_\pm^\times$, defined in (59), is continuous with the topology on $\boldsymbol{\Phi}_\pm^\times$ [26].

Consequently,

H is continuous on $\boldsymbol{\Phi}_\pm$ and on $\boldsymbol{\Phi}_\pm^\times$ but not in $\mathscr{H}$.[19]

(iii) If $t > 0$ and $\psi_+ \in \boldsymbol{\Phi}_+$, then, [26]

$$\mathrm{e}^{\mathrm{i}tH}\,\psi_+ \in \boldsymbol{\Phi}_+ \ . \tag{60}$$

Furthermore, for $t > 0$, $\mathrm{e}^{\mathrm{i}tH}$ is a continuous operator on $\boldsymbol{\Phi}_+$. However, for each $t_0 < 0$, we have a vector $\psi_+^0 \boldsymbol{\Phi}_+$ such that $\mathrm{e}^{\mathrm{i}t_0H}\,\psi_+^0$ does not belong to $\boldsymbol{\Phi}_+$. Consequently, for all $t < 0$, the operator $\mathrm{e}^{\mathrm{i}tH}$

[18] We recall that $\mathscr{H} = V_\pm^{-1}L^2(\mathbb{R}^+)$.

[19] Because its spectrum is not bounded.

is *not* an operator on $\boldsymbol{\Phi}_+$. Using the duality relation (26) we have that, for $t > 0$,

$$\langle e^{itH}\psi_+|F_+\rangle = \langle\psi_+|e^{-itH}F_+\rangle, \quad \forall\psi_+ \in \boldsymbol{\Phi}_+, \ \forall F_+ \in \boldsymbol{\Phi}_+^\times \tag{61}$$

and the evolution operator e^{-itH} is well defined and continuous on $\boldsymbol{\Phi}_+^\times$ for the $t > 0$ *only*. For $t < 0$ e^{-itH} is not even defined on $\boldsymbol{\Phi}_+^\times$.

If $t < 0$ and $\psi_- \in \boldsymbol{\Phi}_-$,

$$e^{itH}\psi_- \in \boldsymbol{\Phi}_- \tag{62}$$

and is a continuous operator on $\boldsymbol{\Phi}_-$. Then, using the duality formula (26), we have that e^{-itH} is well defined and continuous on $\boldsymbol{\Phi}_-^\times$ for $t < 0$. However, for $t > 0$, e^{-itH} is not even defined on $\boldsymbol{\Phi}_-^\times$. These properties follow from the properties of $e^{it\mathscr{E}}$ on $S \cap \mathscr{H}_\pm^2|_{\mathbb{R}^+}$ [26].

(iv) For each $z_+ \in \mathbb{C}^+$, there exists an anti-linear functional $F_z \in \boldsymbol{\Phi}_+^\times$, which is defined as follows: take $\psi_+ \in \boldsymbol{\Phi}_+$, we have that

$$V_+\psi_+ = \psi_+(E) \in S \cap \mathscr{H}_\pm^2|_{\mathbb{R}^+} \ ,$$

and the function $\psi_+(E)$ determines a unique function in $S \cap \mathscr{H}_+^2$. This function is precisely $\theta_+^{-1}\psi_+(E)$, where θ_+^{-1} is the inverse of the mapping θ_+ defined in (49) and (50).

Observe that the functions $\psi_+(E)$ and $\theta_+^{-1}\psi_+(E)$ are boundary values of the same Hardy function $\psi_+(z)$ on the upper half-plane. The difference between them lies on the fact that $\psi_+(E)$ represents the boundary values of $\psi_+(z)$ on the positive semi-axis, $R^+ = [0, \infty)$ and $\theta_+^{-1}\psi_+(E)$ represents the boundary values of $\psi_+(z)$ on the whole real axis $\mathbb{R} = (-\infty, \infty)$. On $\mathbb{R}^+$, both, $\psi_+(E)$ and $\theta_+^{-1}\psi_+(E)$ coincide. Therefore, $\psi_+(E)$ and $\theta_+^{-1}\psi_+(E)$ represent the same function on different domains of definition, $\mathbb{R}^+$ and $\mathbb{R}$, respectively. Then, it seems reasonable to use the symbol $\psi_+(E)$ for both, $\psi_+(E)$ and $\theta_+^{-1}\psi_+(E)$.

Then, let us define the function $\psi_-^\#(E)$ as

$$\psi_-^\#(E) = [\psi_+(E)]^* \ , \tag{63}$$

where, as usual, the star denotes complex conjugation. By the properties of Hardy functions, $\psi_-^\#(E)$ is a Hardy function on the *lower* half-plane. Therefore, it takes values on $\mathbb{C}^-$. As $z_+ \in \mathbb{C}^+$, its complex conjugate $z_+^* \in \mathbb{C}^-$ and we define

$$F_z(\psi_+) = \psi_-^\#(z_+^*) \ . \tag{64}$$

It is not difficult to show that F_z is an anti-linear mapping from $\boldsymbol{\Phi}_+$ into the set of complex numbers $\mathbb{C}$. We can also show that F_z is continuous [26] and, therefore, it belongs to $\boldsymbol{\Phi}_+^\times$. Following the Dirac notation [44], Eq. (63) can be written as

$$F_z(\psi_+) = \langle\psi_+|F_z\rangle \quad \text{or also} \quad F_z(\psi_+) = \langle\psi_+|z_+\rangle \ . \tag{65}$$

In the second equation in (65) the functional F_z has been denoted as $|z_+\rangle$.

For each $z \in \mathbb{C}^-$, we can define a functional $|z_-\rangle$ on $\boldsymbol{\Phi}_-$ in the same manner;

$$\psi_+^\#(E) = [\psi_-(E)]^* \ ,$$

where $\psi_-(E) = V_-\psi_-$ with $\psi_- \in \boldsymbol{\Phi}_-$, and

$$\langle \psi_- | z_- \rangle = \psi_+^{\#}(z_-^*) \ .$$

Here, $|z_-\rangle \in \boldsymbol{\Phi}_-^\times$.

In our case, we have two resonance poles at the points $z_R = E_R - i\Gamma$ and $z_R^* = E_R + i\Gamma$. Following the notation of [5], we shall denote the corresponding states as[20]

$$|z_R\rangle = |f_0\rangle \quad \text{and} \quad |z_R^*\rangle = |\tilde{f}_0\rangle \ . \tag{66}$$

Therefore, the vectors $|f_0\rangle$ and $|\tilde{f}_0\rangle$, fulfill, respectively:

$$\langle \psi_+ | f_0 \rangle = [\psi_+^{\#}(z_0)], \quad \langle \psi_- | \tilde{f}_0 \rangle = [\psi_-^{\#}(z_0^*)] \tag{67}$$

with $\psi_\pm \in \boldsymbol{\Phi}_\pm$.

The functionals $|f_0\rangle \in \boldsymbol{\Phi}_+$ and $|\tilde{f}_0\rangle \in \boldsymbol{\Phi}_-$ are, respectively, the *decaying and growing Gamow vectors.*[21]

Two intermediate spaces are also important. We define them as

$$\boldsymbol{\Psi}_\pm = U^{-1}[S \cap \mathscr{H}_\pm^2|_{\mathbb{R}^+}] \ , \tag{68}$$

where U diagonalizes H_0. The spaces $\boldsymbol{\Psi}_\pm$ have the following properties:

(i) $H_0\boldsymbol{\Psi}_\pm \subset \boldsymbol{\Psi}_\pm$.

(ii) We endow $\boldsymbol{\Psi}_\pm$ with the topology transported from $[S \cap \mathscr{H}_\pm^2|_{\mathbb{R}^+}]$ by U^{-1}. With this topology, H_0 is continuous on $\boldsymbol{\Psi}_\pm$. Then, H_0 can be extended into the antiduals $\boldsymbol{\Psi}_\pm^\times$ by the duality formula (see (26)):

$$\langle H_0\psi_\pm | F_\pm \rangle = \langle \psi_\pm | H_0 F_\pm \rangle, \quad \forall \psi_\pm \in \boldsymbol{\Psi}_\pm, \ F_\pm \in \boldsymbol{\Psi}_\pm^\times \ ,$$

(iii) There is a relation between $\boldsymbol{\Phi}_\pm$ and $\boldsymbol{\Psi}_\pm$ which is given by

$$\boldsymbol{\Phi}_+ = \boldsymbol{\Omega}_{\mathrm{OUT}}\boldsymbol{\Psi}_+, \quad \boldsymbol{\Phi}_- = \boldsymbol{\Omega}_{\mathrm{IN}}\boldsymbol{\Psi}_- \ . \tag{69}$$

These formulas appear as a consequence of (22). The operators $\boldsymbol{\Omega}_{\mathrm{OUT}}$ and $\boldsymbol{\Omega}_{\mathrm{IN}}$ are continuous mappings between $\boldsymbol{\Psi}_\pm$ and $\boldsymbol{\Phi}_\pm$. Since $\boldsymbol{\Omega}_{\mathrm{OUT}}$ and $\boldsymbol{\Omega}_{\mathrm{IN}}$ are unitary on $\mathscr{H}$[22], they can be extended to be

[20] The meaning of the subindex zero will be clarified as we introduce the multiple pole resonances. See [5].

[21] In references [20,26,52] the signs are changed. We prefer to use here the sign plus for the events after the scattering and the sign minus for the events before the scattering, which is more intuitive. However, states in $\boldsymbol{\Phi}_-^\times$ do not represent necessarily situations before the scattering as we shall discuss later.

[22] Or between the scattering states of H_0 and H, if they have bound states.

continuous mappings between the dual spaces [26], defining

$$\langle \boldsymbol{\Omega}_{\mathrm{OUT}}\psi_+|\boldsymbol{\Omega}_{\mathrm{OUT}}F_+\rangle = \langle \psi_+|F_+\rangle, \quad \langle \boldsymbol{\Omega}_{\mathrm{IN}}\psi_-|\boldsymbol{\Omega}_{\mathrm{IN}}F_-\rangle = \langle \psi_-|F_-\rangle \ , \tag{70}$$

where $\psi_\pm \in \boldsymbol{\Psi}_\pm$ and $F_\pm \in \boldsymbol{\Psi}_\pm^\times$. Observe that $\boldsymbol{\Omega}_{\mathrm{IN}}F_- \in \boldsymbol{\Phi}_-^\times$ and $\boldsymbol{\Omega}_{\mathrm{OUT}}F_+ \in \boldsymbol{\Phi}_+^\times$.

The relation between these spaces can be summarized using the following diagrams:

$$\begin{array}{ccccc}
 & \boldsymbol{\Phi}_- & \subset & \mathscr{H}_{\mathrm{ac}}(H) & \subset & \boldsymbol{\Phi}_-^\times \\
\boldsymbol{\Omega}_{\mathrm{IN}}^{-1} & \downarrow & & \downarrow \boldsymbol{\Omega}_{\mathrm{IN}}^{-1} & & \downarrow \boldsymbol{\Omega}_{\mathrm{cIN}}^{-1} \\
 & \boldsymbol{\Psi}_- & \subset & \mathscr{H}_{\mathrm{ac}}(H_0) & \subset & \boldsymbol{\Psi}_-^\times \\
U & \downarrow & & \downarrow U & & \downarrow U \\
 & S\cap\mathscr{H}_-^2|_{\mathbb{R}^+} & \subset & L^2(\mathbb{R}^+) & \subset & (S\cap\mathscr{H}_-^2|_{\mathbb{R}^+})^\times \\
\theta_-^{-1} & \downarrow & & & & \downarrow (\theta_-^{-1})^\times \\
 & S\cap\mathscr{H}_-^2 & \subset & L^2(\mathbb{R}) & \subset & (S\cap\mathscr{H}_-^2)^\times
\end{array} \tag{71}$$

The operator U can be extended to be a continuous operator between $\boldsymbol{\Psi}_\pm^\times$ and $(S\cap\mathscr{H}_\pm^2|_{\mathbb{R}^+})^\times$, also by a duality formula. Note that θ_-^{-1} does not connect $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R})$, because θ_-^{-1} originates in a property of Hardy functions which is not shared by L^2 functions. As $\theta_\pm$ are not unitary, their adjoints $\theta_\pm^\times$, defined in (50), do not extend $\theta_\pm$ [26]. Note that $(\theta_\pm^\times)^{-1} = (\theta_\pm^{-1})^\times$, a property which can be obtained from definitions (50). The second diagram is

$$\begin{array}{ccccc}
 & \boldsymbol{\Phi}_+ & \subset & \mathscr{H}_{\mathrm{ac}}(H) & \subset & \boldsymbol{\Phi}_+^\times \\
\boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} & \downarrow & & \downarrow \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} & & \downarrow \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} \\
 & \boldsymbol{\Psi}_+ & \subset & \mathscr{H}_{\mathrm{ac}}(H_0) & \subset & \boldsymbol{\Psi}_+^\times \\
U & \downarrow & & \downarrow U & & \downarrow U \\
 & S\cap\mathscr{H}_+^2|_{\mathbb{R}^+} & \subset & L^2(\mathbb{R}^+) & \subset & (S\cap\mathscr{H}_+^2|_{\mathbb{R}^+})^\times \\
\theta_+^{-1} & \downarrow & & & & \downarrow (\theta_+^{-1})^\times \\
 & S\cap\mathscr{H}_+^2 & \subset & L^2(\mathbb{R}) & \subset & (S\cap\mathscr{H}_+^2)^\times
\end{array} \tag{72}$$

This diagram is self-explanatory.

3.2. Properties of Gamow vectors

The Gamow vectors $|f_0\rangle$ and $|\tilde{f}_0\rangle$ have the following properties:

(i) They are generalized eigenvectors of the Hamiltonian, H, and satisfy the spectral equations [20,26,52,5]:

$$H|f_0\rangle = z_R|f_0\rangle, \quad H|\tilde{f}_0\rangle = z_R^*|\tilde{f}_0\rangle \ . \tag{73}$$

Bound states of H represent physical states that do not evolve under the action of H. Their energies are real. They satisfy a spectral equation of the form $H\varphi = E\varphi$, with E real. In the language that we are currently using, bound states are stable because their half-life, defined as the inverse of the imaginary part of the eigenvalue, is infinite [25]. On the other hand, decaying states, satisfy an analogous spectral equation with complex eigenvalues.

Bound states represent *stable* states and Gamow vector represent *unstable* states. Both, stable and unstable states satisfy similar spectral equations and they differ in the character of the eigenvalues [20].

The idea that resonances are eigenvectors of H with eigenvalues at the resonance poles is due to Nakanishi [88].

(ii) We know that the action of the evolution operator e^{-itH} on $|f_0\rangle$ is valid for $t > 0$, and for $|\tilde{f}_0\rangle$ for $t < 0$. Taking into account these restrictions, we can prove that [26]:

$$e^{-itH}|f_0\rangle = e^{-itE_R}e^{-t\Gamma}|f_0\rangle, \quad t > 0 \ , \tag{74}$$

$$e^{-itH}|\tilde{f}_0\rangle = e^{-itE_R}\, e^{\Gamma t}|\tilde{f}_0\rangle, \quad t < 0 \ . \tag{75}$$

It means that the Gamow vector $|f_0\rangle$ decays exponentially for $t > 0$ while $|\tilde{f}_0\rangle$ grows exponentially from $t = -\infty$ to $t = 0$. The vector state $|f_0\rangle$ would represent the state of an exponentially decaying system, like the system formed by unstable particles. The growing process described by (75) is the mirror image of the decaying process (74). The growing process usually does not describe the capture process or the process of creation of a resonance, because these two processes are asymmetric [77].

(iii) If T represents the time reversal operator, we have that [23,58,36]

$$T|f_0\rangle = |\tilde{f}_0\rangle, \quad T|\tilde{f}_0\rangle = |f_0\rangle \ . \tag{76}$$

Time reversal transforms the decaying process in its mirror image and vice versa.

3.3. Degenerate resonances

The term *degenerate resonances* has been coined by Mondragón et al. [86] to denote resonances that are represented by multiple poles of the analytic continuation of the scattering function $S(E)$. They are poles on the second Riemann sheet. We shall see, in the next subsection, the existing relation between these degenerate resonances and a new type of Gamow vectors. Also, we shall show how to attach to any resonant pole of order N, exactly $N+1$ Gamow vectors of the new type.

Here, we shall present the mathematical construction of these new Gamow vectors and some of their properties.

Let $\phi_\pm \in \boldsymbol{\Phi}_\pm$ and $\phi_\pm(E) = V_\pm\phi_\pm$, defined as in Section 3.1. We recall that $\phi_\pm^\#(E) = [\phi_\pm(E)]^*$ and that $\phi_\pm^\#(E) \in S \cap \mathscr{H}_\mp|_{\mathbb{R}^+}$. Let z_R be a pole of $S(E)$ in the lower half-plane of the second Riemann

sheet. Then, let us consider the following mappings:

$$\phi_+ \mapsto \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,\phi_-^{\#}(z)\right]_{z=z_{\mathrm{R}}}, \quad \phi_- \mapsto \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,\phi_+^{\#}(z)\right]_{z=z_{\mathrm{R}}^*}, \tag{77}$$

with all values of n. If $n=0$ these maps represent the Gamow functionals $|f_0\rangle$ and $|\tilde{f}_0\rangle$, respectively. For all values of n, we shall denote them as

$$|f_n\rangle \quad \text{and} \quad |\tilde{f}_n\rangle\,, \tag{78}$$

such that

$$\langle\phi|f_n\rangle = \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,\phi_-^{\#}(z)\right]_{z=z_{\mathrm{R}}} \quad \text{and} \quad \langle\phi_-|\tilde{f}_n\rangle = \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,\phi_+^{\#}(z)\right]_{z=z_{\mathrm{R}}^*}. \tag{79}$$

We can show that the functionals (77) are continuous and antilinear on $\Phi_\pm$ [5]. Moreover, we can prove the following formulas [5]:

$$H|f_n\rangle = z_{\mathrm{R}}|f_n\rangle + n|f_{n-1}\rangle \tag{80}$$

and

$$H|\tilde{f}_n\rangle = z_{\mathrm{R}}^*|\tilde{f}_n\rangle + n|\tilde{f}_{n-1}\rangle\,, \tag{81}$$

valid for $n=0,1,2,\ldots,N-1$, where N is the multiplicity of the poles[23] z_{R} and z_{R}^*. Note that for $n=0$, we recover (73). Let us define the Nth-dimensional subspace $\Phi_{N+}^{\times}$ of $\Phi_+^{\times}$ as the subspace spanned by the vectors $|f_j\rangle$, $j=0,1,\ldots,N-1$. Similarly, we define the Nth-dimensional subspace $\Phi_{N-}^{\times}$ as the subspace of Φ spanned by the vectors $|\tilde{f}_j\rangle$, $j=0,1,\ldots,N-1$. We have that

$$H\Phi_{N\pm}^{\times} \subset \Phi_{N\pm}^{\times}\,. \tag{82}$$

On $\Phi_{N+}^{\times}$, we see after (80) that in the basis given by $|f_0\rangle, |f_1\rangle, \ldots, [1/(N-2)]|f_{N-2}\rangle$, $[1/(N-1)]|f_{N-1}\rangle$, H admits the block diagonal representation:

$$H = \begin{pmatrix} z_{\mathrm{R}} & 1 & 0 & \ldots & 0 \\ 0 & z_{\mathrm{R}} & 1 & \ldots & 0 \\ 0 & 0 & z_{\mathrm{R}} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & z_{\mathrm{R}} \end{pmatrix} \tag{83}$$

On $\Phi_{N-}^{\times}$, we see after (81) that in the basis $|\tilde{f}_0\rangle, |\tilde{f}_1\rangle, \ldots, [1/(N-2)]|\tilde{f}_{N-2}\rangle$, $[1/(N-1)]|\tilde{f}_{N-1}\rangle$, H admits the following block diagonal representation:

$$H = \begin{pmatrix} z_{\mathrm{R}}^* & 1 & 0 & \ldots & 0 \\ 0 & z_{\mathrm{R}}^* & 1 & \ldots & 0 \\ 0 & 0 & z_{\mathrm{R}}^* & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & z_{\mathrm{R}}^* \end{pmatrix}. \tag{84}$$

[23] Causality relations show that both z_{R} and z_{R}^* must have the same multiplicity [90].

The evolution law of these Gamow vectors is given by [5]

$$e^{-itH}|f_n\rangle = e^{-itz_R}\sum_{l=0}^{N-1}\binom{N-1}{l}(-it)^l|f_{n-l}\rangle\ , \tag{85}$$

which is valid only for $t<0$, and

$$e^{-itH}|\tilde{f}_n\rangle = e^{itz_R^*}\sum_{l=0}^{N-1}\binom{N-1}{l}(it)^l|\tilde{f}_{n-l}\rangle\ , \tag{86}$$

valid only for $t<0$.

From (85) and (86), we observe that only $|f_0\rangle$ and $|\tilde{f}_0\rangle$ evolve exponentially. For all the other vectors $|f_0\rangle$ and $|\tilde{f}_0\rangle$, the exponential evolution law is multiplied by a polynomial of t of degree N. On $\boldsymbol{\Phi}_{N+}^{\times}$, the evolution operator can be written in matrix form as

$$e^{-itH} = e^{-itz_R}\begin{pmatrix} 1 & -it & \frac{(-it)^2}{2!} & \dots & \frac{(-it)^{N-1}}{(N-1)!} \\ 0 & 1 & -it & \dots & \frac{(-it)^{N-2}}{(N-2)!} \\ 0 & 0 & 1 & \dots & \frac{(-it)^{N-3}}{(N-3)!} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad t>0\ . \tag{87}$$

Analogously, the matrix form of the evolution operator on $\boldsymbol{\Phi}_{N-}^{\times}$ is

$$e^{-itH} = e^{itz_R^*}\begin{pmatrix} 1 & it & \frac{(it)^2}{2!} & \dots & \frac{(it)^{N-1}}{(N-1)!} \\ 0 & 1 & it & \dots & \frac{(it)^{N-2}}{(N-2)!} \\ 0 & 0 & 1 & \dots & \frac{(it)^{N-3}}{(N-3)!} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad t<0\ . \tag{88}$$

3.4. Resonance scattering and Gamow vectors

In this subsection, we shall explain how Gamow vectors arise in a resonant scattering process, as it was suggested by Bohm [20]. We assume that resonances are produced in a resonant scattering process. A resonance scattering process implies the existence of a *free* dynamics, governed by a *free* Hamiltonian H_0, and a *perturbed* dynamics governed by the Hamiltonian $H = H_0 + V$. In the distant past, we prepare a state $\psi^{\text{in}}(t)$ that evolves freely, under the action of H_0, until it enters in the interaction region where it feels the action of the potential V. After a time, it goes out of the

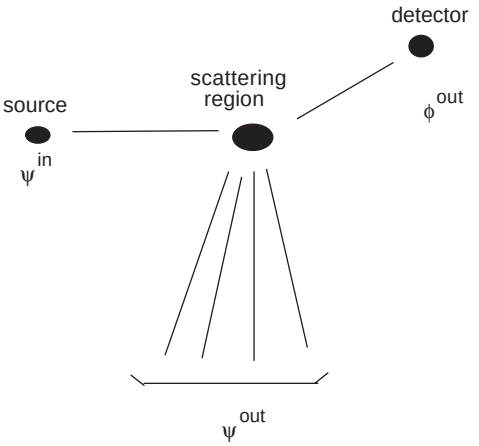

Fig. 1. Schematic representation of a scattering process (see Section 3.4). The state $\psi^{(\mathrm{in})}$ evolves freely, from the source, until it reaches the interaction region. $\psi^{(\mathrm{out})}$ is the outgoing state and $\phi^{(\mathrm{out})}$ is its projection onto the measurement apparatus, i.e. it is the observed state.

interaction region and evolves freely again as $\psi^{\mathrm{out}}(t)$. Fig. 1 shows, schematically, the scattering process we are interested in. The relation between the incoming free state ψ^{in} and the outgoing free state ψ^{out} is given by the S operator:

$$\psi^{\mathrm{out}}(t) = S\psi^{\mathrm{in}}(t) \ . \tag{89}$$

If the particle represented by this quantum state spends, within the interaction region, a time *much larger* than the time it would spent if the interaction would not exists, we say that a *quasi-stationary state or a resonance* has been produced. This may happen for certain values of the energy, i.e. at the resonant energies of the prepared state. Under certain rather general conditions [20], this resonance is characterized by a pair of poles in the analytic continuation of the S operator. If we work in the momentum representation, the function $S(k)$ is *mesomorphic* in the complex plane ($k \in \mathbb{C}$). The resonant poles are located in the lower open half plane ($\mathrm{Im}\, k < 0$) and are symmetric with respect to the imaginary axis. If we work in the energy representation, the resonance poles of $S(E)$ are located in the second sheet of the Riemann surface at $z_{\mathrm{R}} = E_{\mathrm{R}} - \mathrm{i}\Gamma/2$ and $z_{\mathrm{R}}^* = E_{\mathrm{R}} + \mathrm{i}\Gamma/2$. Here E_{R} represents the resonant energy of the quasi-stationary state and $\hbar/\Gamma$ is the half-life. Fig. 2 shows the location of the resonant pole in the complex plane. We have to stress that, in general, we do not observe the outgoing state $\psi^{\mathrm{out}}(t)$ but, instead, its projection into the region where the measurement apparatus is placed. Therefore, we need to project $\psi^{\mathrm{out}}(t)$ into this region to obtain the observed state $\phi^{\mathrm{out}}(t)$.

The idea of Bohm is to calculate the transition amplitude between the true outgoing state and the observed outgoing state. This gives

$$\langle \phi^{\mathrm{out}}(t) | \psi^{\mathrm{out}}(t) \rangle = \langle \phi^{\mathrm{out}} | S\psi^{\mathrm{in}} \rangle \ . \tag{90}$$

Now, let us assume that the Møller wave operators exist for the pair (H_0, H) and that asymptotic completeness holds. Then,

$$S = \boldsymbol{\Omega}_{\mathrm{OUT}}^{\dagger} \boldsymbol{\Omega}_{\mathrm{IN}}$$

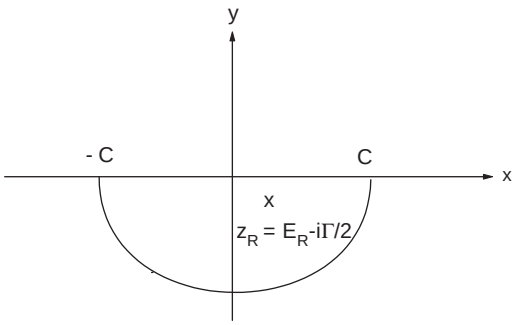

Fig. 2. Resonance poles of the S-operator.

and (90) yields

$$\langle \phi^{\text{out}} | \boldsymbol{\Omega}^{\dagger}_{\text{OUT}} \boldsymbol{\Omega}_{\text{IN}} \psi^{\text{in}} \rangle = \langle \boldsymbol{\Omega}_{\text{OUT}} \phi^{\text{out}} | \boldsymbol{\Omega}_{\text{IN}} \psi^{\text{in}} \rangle = \langle \phi_+ | \psi_- \rangle \ , \tag{91}$$

where obviously

$$\phi_+ = \boldsymbol{\Omega}_{\text{OUT}} \phi^{\text{out}}, \quad \psi_- = \boldsymbol{\Omega}_{\text{IN}} \psi^{\text{in}} \ . \tag{92}$$

Since we are in the situation described in Section 3.1, Eq. (90) gives

$$\langle \phi^{\text{out}} | S \psi^{\text{in}} \rangle = \int_0^{\infty} [\phi^{\text{out}}(E)]^* S(E) \psi^{\text{in}}(E) \, \mathrm{d}E \ . \tag{93}$$

Let us make now the following ansatz:

$$\phi^{\text{out}} \in \boldsymbol{\Psi}_+, \quad \psi^{\text{in}} \in \boldsymbol{\Psi}_- \ , \tag{94}$$

where $\boldsymbol{\Psi}_{\pm}$ have been defined in (68). The operator U diagonalizes the free Hamiltonian H_0. Therefore, it connects the abstract free evolving vectors with their wave functions in the energy representation. Thus, $\phi^{\text{out}}(E) = U\phi^{\text{out}}$ and $\psi^{\text{in}}(E) = U\psi^{\text{in}}$. Hence,

$$\phi^{\text{out}}(E) \in S \cap \mathscr{H}^2_+|_{\mathbb{R}^+}, \quad \psi^{\text{in}}(E) \in S \cap \mathscr{H}^2_-|_{\mathbb{R}^+} \ . \tag{95}$$

According to the theorem by Gelfand and Maurin mentioned in Section 2.6, there exist eigenvectors of H_0 in $\boldsymbol{\Psi}^{\times}_{\pm}$, such that $H_0|E\rangle = E\,|E\rangle$, for all $E \in \mathbb{R}^+$ [59], with the following property [26]:

$$\langle \psi^{\text{in}} | E \rangle = [\psi^{\text{in}}(E)]^* \in S \cap \mathscr{H}^2_+|_{\mathbb{R}^+}, \quad \langle \phi^{\text{out}} | E \rangle = [\phi^{\text{out}}(E)]^* \in S \cap \mathscr{H}^2_-|_{\mathbb{R}^+} \ , \tag{96}$$

so that

$$\langle \phi^{\text{out}} \,|\, S \psi^{\text{in}} \rangle = \int_0^{\infty} \langle \phi^{\text{out}} | E \rangle S(E) \langle E | \psi^{\text{in}} \rangle \, \mathrm{d}E \ , \tag{97}$$

where

$$\langle E | \psi^{\text{in}} \rangle = \langle \psi^{\text{in}} | E \rangle^* \ . \tag{98}$$

Let us define now

$$|E_+\rangle = \boldsymbol{\Omega}_{\text{OUT}} |E\rangle, \quad |E_-\rangle = \boldsymbol{\Omega}_{\text{IN}} |E\rangle \ . \tag{99}$$

The functionals $|E_\pm\rangle \in \boldsymbol{\Phi}_\pm^\times$ have the following properties:

$$H|E_+\rangle = H\boldsymbol{\Omega}_{\rm OUT}|E\rangle = \boldsymbol{\Omega}_{\rm OUT}H_0|E\rangle = E\boldsymbol{\Omega}_{\rm OUT}|E\rangle = E|E_+\rangle,$$
$$H|E_-\rangle = H\boldsymbol{\Omega}_{\rm IN}|E\rangle = \boldsymbol{\Omega}_{\rm IN}H_0|E\rangle = E\boldsymbol{\Omega}_{\rm IN}|E\rangle = E|E_-\rangle\ . \tag{100}$$

There is one pair of non-degenerate states $|E_\pm\rangle$ for each $E \in \mathbb{R}^+$, the spectrum of H [24] Therefore, the theorem by Gelfand and Maurin applies here so that $\langle\psi_-|E_-\rangle$ and $\langle\phi_+|E_+\rangle$ are the complex conjugate of the wave function $\psi_-(E)$ and $\phi_+(E)$ for the state vectors ψ_- and ϕ_+ in the energy representation. Also, we have that

$$\langle\phi_+|E_+\rangle = \langle\boldsymbol{\Omega}_{\rm OUT}\phi^{\rm out}|\boldsymbol{\Omega}_{\rm OUT}|E\rangle = \langle\phi^{\rm out}|E\rangle = [\phi^{\rm out}(E)]^* \tag{101}$$

and

$$\langle\psi_-|E_-\rangle = \langle\boldsymbol{\Omega}_{\rm IN}\psi^{\rm in}|\boldsymbol{\Omega}_{\rm IN}|E\rangle = \langle\psi^{\rm in}|E\rangle = [\psi^{\rm in}(E)]^*\ . \tag{102}$$

Now, in order to obtain the Gamow vectors, we proceed as follows: first, let us consider the scalar product

$$\langle\phi_+|\psi_-\rangle = \langle\phi^{\rm out}|S\psi^{\rm in}\rangle = \int_0^\infty \langle\phi^{\rm out}|E\rangle S(E)\langle E|\psi^{\rm in}\rangle\,{\rm d}E$$
$$= \int_0^\infty [\phi^{\rm out}(E)]^* S(E)\psi^{\rm in}(E)\,{\rm d}E\ . \tag{103}$$

The function $[\phi^{\rm out}(E)]^*\psi^{\rm in}(E)$ is analytic on the lower open ($\mathrm{Im}\,E < 0$) half-plane, where the function $S(E)$ is analytic except for isolated singularities. [25] Let us assume that $S(E)$ contains resonance poles only. Using the Cauchy theorem [26,52], we obtain

$$\int_0^R [\phi^{\rm out}(E)]^* S(E)\psi^{\rm in}(E)\,{\rm d}E = -\int_{-R}^0 [\phi^{\rm out}(E)]^* S(E)\psi^{\rm in}(E)\,{\rm d}E$$
$$+ \int_C [\phi^{\rm out}(z^*)]^* S(z)\psi^{\rm in}(z)\,{\rm d}z$$
$$- 2\pi{\rm i}\sum \text{Residues}\{[\phi^{\rm out}(z)]^* S(z)\psi^{\rm in}(z)\}\ , \tag{104}$$

where

(i) The integral over the negative axis refers to the negative axis in the second sheet of the Riemann surface.

(ii) C is the semicircle, in the lower half-plane of the second sheet, centered at the origin with radius R, which does not contain any pole of $S(E)$.

---

[24] The non-degeneracy holds in the case being considered here, for a spherically symmetric potential and for $l = 0$. For the general case $|E_\pm\rangle$ are degenerate, but this does not affect the discussion.

[25] If we use the language of Riemann surfaces, we note that the function $[\phi^{\rm out}(E)]^*\psi^{\rm in}(E)$ has to be defined on the upper rim of the cut, of the surface associated to the transformation $z \mapsto \sqrt{z}$, that connects the upper half-plane of the first sheet with the lower half-plane of the second sheet. This second sheet contains the resonance poles [20]. The geometry is the one shown in Fig. 3.

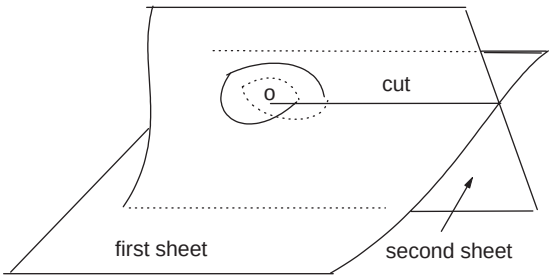

Fig. 3. Geometry of the Riemann surfaces (first and second sheets) and the location of the resonance poles, as explained in Section 3.4.

(iii) The sum of the residues extends over all poles of $S(E)$ in the region limited by the contour $[-R,R]\cup C$.

Fig. 3 illustrates these concepts. If we take the limit as $R\mapsto\infty$ and assume that $S(E)$ is bounded by a polynomial on the second sheet (see (56)), the integral over C goes to zero and we obtain

$$\int_0^\infty [\phi^{\rm out}(E)]^* S(E)\psi^{\rm in}(E)\,{\rm d}E = -\int_{-\infty}^0 [\phi^{\rm out}(E)]^* S(E)\psi^{\rm in}(E)\,{\rm d}E$$
$$-2\pi{\rm i}\sum \text{Residues}\{[\phi^{\rm out}(z^*)]^* S(z)\psi^{\rm in}(z)\}\ . \tag{105}$$

The sum includes the residues of all the poles in the lower half-plane of the second sheet. The number of these poles is infinite for realistic models such as those with spherically symmetric potentials. However, we may propose toy resonant models that are completely solvable and that give us an excellent picture of the behavior of resonances. We can construct toy models having one resonance only. Examples of those are the Friedrichs model, to be discussed in the next section, and some completely solvable one-dimensional models. In those models we assume "a priori" the form of $S(k)$ with assigned resonance poles and then, obtain the potential [8,40]. Thus, we may assume, for the sake of simplicity, the existence of a resonance pole located at $z_{\rm R}$ and that this pole is simple. In this case, the sum of the residues in (105) has one term, which is

$$-2\pi{\rm i}[\phi^{\rm out}(z_{\rm R}^*)]^* {\rm s}_1\,\psi^{\rm in}(z_{\rm R}) = -2\pi{\rm i}\langle\phi_+|f_0\rangle\langle\tilde f_0|\psi_-\rangle\ , \tag{106}$$

where s_1 is the residue of $S(z)$ at $z_{\rm R}$.

Then, using (105) and (106), we found that[26]

$$\langle\phi_+|\psi_-\rangle = \int_{-\infty}^0 \langle\phi_+|E_+\rangle S(E)\langle E_-|\psi_-\rangle\,{\rm d}E - 2\pi{\rm i}\langle\phi_+|f_0\rangle {\rm s}_1\langle\tilde f_0|\psi_-\rangle$$
$$= \text{background} - 2\pi{\rm i}\langle\phi_+|f_0\rangle {\rm s}_1\langle\tilde f_0|\psi_-\rangle\ , \tag{107}$$

where "background" represents the integral term in (107). If we define the complex constant $A=-2\pi{\rm i}s_1\langle\tilde f_0|\psi_-\rangle$, which obviously depends on ψ_-, this background term is written as

$$\text{background} = \langle\phi_+,\psi_-\rangle - A\langle\phi_+|f_0\rangle\ . \tag{108}$$

[26] For $E<0$, the mappings $\psi^{\rm in}\mapsto\langle\psi^{\rm in}|E\rangle=[\psi^{\rm in}(E)]^*$, $\phi_+\mapsto\langle\phi_+|E_+\rangle=\langle\boldsymbol{\Omega}_{\rm OUT}\phi_+|\boldsymbol{\Omega}_{\rm OUT}|E_+\rangle=\langle\phi^{\rm out}|E\rangle=[\phi^{\rm in}(E)]^*$, etc., are also antilinear and continuous.

Since, $\psi_- \in \mathscr{H} \subset \boldsymbol{\Phi}_+^\times$, the vector ψ_- can be identified to a continuous anti-linear functional on $\boldsymbol{\Phi}_+$, for which the action on each $\phi_+ \in \boldsymbol{\Phi}_+$ is the scalar product $\langle \phi_+ | \psi_- \rangle$. Therefore, the background term in (102) is given by the action of a functional on ϕ_+. This functional is $\psi_- - A|f_0\rangle$ and we call it $|\text{bgk}\rangle$ for simplicity, so that

$$\text{background} = \langle \phi_+ | \text{bgk} \rangle \ .$$

If we omit the arbitrary vector $\phi_+ \in \boldsymbol{\Phi}_+$ in (107), we have

$$\psi_- = |\text{bgk}\rangle + A\,|f_0\rangle \ . \tag{109}$$

Thus, the decaying state ψ_- is a sum of two terms, $A\,|f_0\rangle$ proportional to the Gamow vector $|f_0\rangle$, that decays exponentially, and the "background" functional $|\text{bgk}\rangle$. The background functional $|\text{bgk}\rangle$ has the following properties:

(i) The state $|\text{bgk}\rangle$ decays to the future in the sense that

$$\langle \phi_+ | \mathrm{e}^{-\mathrm{i}tH} | \text{bgk} \rangle \mapsto 0, \quad t \mapsto 0^+, \quad \forall \phi_+ \in \boldsymbol{\Phi}_+ \ . \tag{110}$$

This fact is a consequence of the properties of the integrable functions and, in particular, of the Riemann–Lebesgue lemma [99].

(ii) The state $|\text{bgk}\rangle$ decays slower than an exponential for larger values of t. It is clear after decomposition (109) and from the fact that a normalize vector in $\mathscr{H}$ cannot decay exponentially [50].

Finally, there is a "time reversal" of decomposition (109). This is the decomposition of ϕ_+ in the sum of a vector proportional to $|\tilde{f}_0\rangle$ plus a background type functional [20,26].

3.5. The Friedrichs model

An exactly soluble model for resonance scattering is given by the Friedrichs model [13].

The simplest form of the Friedrichs model [51] includes a free Hamiltonian H_0 with a simple continuous spectrum, which is $\mathbb{R}^+ \equiv [0, \infty)$, plus an eigenvalue ω_0 imbedded in this continuous spectrum ($\omega_0 > 0$). An interaction is produced between the continuous and discrete parts of H_0 by means of a potential V so that the bound state of H_0 is dissolved in the continuous and a resonance is produced. The spectrum of the total Hamiltonian $H = H_0 + V$ is purely continuous and coincides with $\mathbb{R}^+$. Furthermore, the Møller wave operators and, therefore, the S-operator, are well defined in the case of the Friedrichs model [66,47]. The resonance appears as a pole in the analytic continuation of the reduced resolvent as we shall explain next.

From the point of view of the Hilbert space formulation of quantum mechanics, the Hilbert space of this system in the energy representation is the direct sum

$$\mathscr{H} = \mathbb{C} \oplus L^2(\mathbb{R}^+) \ , \tag{111}$$

so that, any $\psi \in \mathscr{H}$ can be represented as

$$\psi = \begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix} \ , \tag{112}$$

where α is a complex number and $\varphi(\omega) \in L^2(\mathbb{R}^+)$. The scalar product of two vectors in $\mathscr{H}$ is given by

$$\left\langle \begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix} \middle| \begin{pmatrix} \beta \\ \eta(\omega) \end{pmatrix} \right\rangle = \alpha^* \beta + \int_0^\infty \varphi^*(\omega)\eta(\omega)\,\mathrm{d}\omega \ . \tag{113}$$

Then, the action of the Hamiltonian H_0 on ψ is

$$H_0\psi = \begin{pmatrix} \omega_0\alpha \\ \omega\varphi(\omega) \end{pmatrix} \ , \tag{114}$$

where $\omega_0 > 0$ is the eigenvalue of H_0 and $\varphi(\omega)$ is a function on $\mathbb{R}^+$ with the property $\omega\varphi(\omega) \in L^2(\mathbb{R}^+)$. Observe that the restriction of H_0 to $L^2(\mathbb{R}^+)$ is the multiplication operator:

$$H_0 \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix} = \omega \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix} \ . \tag{115}$$

This operator has an absolutely continuous spectrum equal to $\mathbb{R}^+$. Thus, H_0 has an eigenvalue, ω_0, imbedded in its continuous spectrum, $\mathbb{R}^+$. The vector

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{116}$$

is the state vector for the bound state. Observe that $H_0|1\rangle = \omega_0|1\rangle$. After the definition of $\mathscr{H}$ in (111), we see that $|1\rangle$ is orthogonal to the space $L^2(\mathbb{R}^+)$.

The total Hamiltonian is given by

$$H = H_0 + \lambda V \tag{117}$$

with

$$V\psi = \begin{pmatrix} \displaystyle\int_0^\infty f(\omega)\,\varphi(\omega)\,\mathrm{d}\omega \\ \alpha f^*(\omega) \end{pmatrix} \ , \tag{118}$$

where, λ is a positive real parameter, the coupling parameter, and $f(\omega)$ is a function in $L^2(\mathbb{R}^+)$ called the *form factor*.

Let us denote by P the orthogonal projection to the subspace spanned by $|1\rangle$ and let Q be its orthogonal complement. Then, Q projects ψ in (112) into its second component. We obviously have that

$$P + Q = I, \quad PQ = QP = O \ , \tag{119}$$

where I and O are the identity and the zero operator on $\mathscr{H}$, respectively.

Proposition F.1. *We have that*

$$QVQ = O \ . \tag{120}$$

Proof. Let us apply QVQ to ψ in (112):

$$QVQ\begin{pmatrix}\alpha\\ \varphi(\omega)\end{pmatrix}=QV\begin{pmatrix}0\\ \varphi(\omega)\end{pmatrix}=Q\begin{pmatrix}\int_0^\infty f(\omega)\varphi(\omega)\,\mathrm{d}\omega\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\ . \qquad \square \tag{121}$$

This is often called the Friedrichs condition [47]. The Friedrichs condition is nothing else than a straightforward consequence of the model.

Proposition F.2. *The projections P and Q commute with H_0.*

Proof. It is sufficient to prove that $PH_0 = H_0P$, since $Q = I - P$.

$$PH_0\begin{pmatrix}\alpha\\ \varphi(\omega)\end{pmatrix}=P\begin{pmatrix}\omega_0\alpha\\ \omega\varphi(\omega)\end{pmatrix}=\begin{pmatrix}\omega_0\alpha\\ 0\end{pmatrix}\ ,$$

$$H_0P\begin{pmatrix}\alpha\\ \varphi(\omega)\end{pmatrix}=H_0\begin{pmatrix}\alpha\\ 0\end{pmatrix}=\begin{pmatrix}\omega_0\alpha\\ 0\end{pmatrix}\ . \tag{122}$$

Under some conditions, that we are going to describe soon, the Friedrichs model has a resonance. The effect of λV is to transform the bound state $|1\rangle$ of H_0 in an unstable state. The real number ω_0 is transformed into a complex number ω_λ that goes to ω_0 as $\lambda \mapsto 0$.

In order to describe resonances, we consider the reduced resolvent of H in $|1\rangle$ given by

$$F_H(z)=\langle 1|\frac{1}{z-H}|1\rangle\ , \tag{123}$$

where H is the total Hamiltonian given by (117). The complex valued function $F_H(z)$ has no singularities on the complex plane other than a branch cut coinciding with the spectrum[27] of H: $\sigma(H)\equiv\mathbb{R}^+$ [66]. Under certain conditions on the form factor $f(\omega)$, $F_H(z)$ admits analytic continuations from above to below and from below to above through the cut. These analytic continuations may have singularities which are associated with the resonances. We shall present here these ideas in detail. To start with, we shall introduce the following theorems:

Theorem F.3. *The reduced resolvent* (123) is given by [47]

$$\langle 1|\,\frac{1}{z-H}\,|1\rangle=\left(-z+\omega_0+\lambda^2\int_0^\infty\frac{|f(\omega)|^2}{z-\omega}\,\mathrm{d}\omega\right)^{-1}\ . \tag{124}$$

Proof. The second resolvent identity [113] states that

$$R(z,H)=R(z,H_0)-\lambda R(z,H_0)VR(z,H)\ , \tag{125}$$

[27] After the definition of the resolvent of H, it is clear why $F_H(z)$ must be singular at the points of the spectrum of H.

where

$$R(z,H)=\frac{1}{z-H},\qquad R(z,H_0)=\frac{1}{z-H_0}\ .$$

This gives

$$P\frac{1}{z-H}P=P\frac{1}{z-H_0}P-\lambda P\frac{1}{z-H_0}V\frac{1}{z-H}P\ . \tag{126}$$

Inserting (119) in the second term in the right of (126), we have:

$$\begin{aligned}P\frac{1}{z-H_0}(P+Q)V(P+Q)\frac{1}{z-H}P&\\ =P\frac{1}{z-H_0}PVP\frac{1}{z-H}P+P\frac{1}{z-H_0}QVP\frac{1}{z-H}P&\\ +P\frac{1}{z-H_0}PVQ\frac{1}{z-H}P+P\frac{1}{z-H_0}QVQ\frac{1}{z-H}P\ .&\end{aligned} \tag{127}$$

As P and Q commute with H_0 and $PQ=O$, we have that

$$P\frac{1}{z-H_0}Q=O\ ,$$

and therefore the second and fourth terms in the right-hand side of (127) vanish. We also have that

$$Q\frac{1}{z-H}P=Q\frac{1}{z-H_0}P-\lambda Q\frac{1}{z-H_0}V\frac{1}{z-H}P\ . \tag{128}$$

The first term of the right-hand side of (128) also vanishes. Then, we have

$$\begin{aligned}Q\frac{1}{z-H}P&=-\lambda Q\frac{1}{z-H_0}(P+Q)V(P+Q)\frac{1}{z-H}P\\ &=-\lambda Q\frac{1}{z-H_0}QVQ\frac{1}{z-H}P-\lambda Q\frac{1}{z-H_0}PVP\frac{1}{z-H}P\\ &\quad-\lambda Q\frac{1}{z-H_0}PVQ\frac{1}{z-H}P-\lambda Q\frac{1}{z-H_0}QVP\frac{1}{z-H}P\ .\end{aligned} \tag{129}$$

Again, the second and the third term in the right-hand side of (129) vanish. Due to the Friedrichs condition (120), also the first term vanishes. Thus,

$$Q\frac{1}{z-H}P=-\lambda Q\frac{1}{z-H_0}QVP\frac{1}{z-H}P\ . \tag{130}$$

Inserting (130) into (127) and then, (127) into (126), we have

$$\begin{aligned}P\frac{1}{z-H}P&=P\frac{1}{z-H_0}P-\lambda P\frac{1}{z-H_0}PVP\frac{1}{z-H}P\\ &\quad+\lambda^2P\frac{1}{z-H_0}VQ\frac{1}{z-H_0}QVP\frac{1}{z-H}P\ .\end{aligned} \tag{131}$$

If we multiply (131) to the left by $(z - H_0)P$, we can write

$$(z - H_0)\,P\,\frac{1}{z - H_0}\,P = P$$

and therefore,

$$P = \left[(z - H_0)P + \lambda PVP - \lambda^2 PVQ\,\frac{1}{z - H_0}\,QVP\right] P\,\frac{1}{z - H}\,P\ , \tag{132}$$

or

$$P\,\frac{1}{z - H}\,P = [G(z)]^{-1}\,P\ , \tag{133}$$

where

$$G(z) = \left[(z - H_0)\,P + \lambda PVP - \lambda^2 PVQ\,\frac{1}{z - H_0}\,QVP\right]\ . \tag{134}$$

Since,

$$\begin{aligned} PVQ\,\frac{1}{z - H_0}\,QVP\begin{pmatrix}1\\0\end{pmatrix} &= PVQ\,\frac{1}{z - H_0}\,QV\begin{pmatrix}1\\0\end{pmatrix} \\ &= PVQ\,\frac{1}{z - H_0}\,Q\begin{pmatrix}0\\ f^*(\omega)\end{pmatrix} = PVQ\,\frac{1}{z - H_0}\begin{pmatrix}0\\ f^*(\omega)\end{pmatrix} \\ &= PVQ\begin{pmatrix}0\\ \dfrac{f^*(\omega)}{z - \omega}\end{pmatrix} = P\begin{pmatrix}\displaystyle\int_0^\infty \frac{|f(\omega)|^2}{z - \omega}\,\mathrm{d}\omega\\ 0\end{pmatrix} \\ &= \begin{pmatrix}\displaystyle\int_0^\infty \frac{|f(\omega)|^2}{z - \omega}\,\mathrm{d}\omega\\ 0\end{pmatrix}\ , \end{aligned} \tag{135}$$

and $PVP|1\rangle = \mathbf{0}$, we have that

$$G(z)|1\rangle = \left[z - \omega_0 - \lambda^2\int_0^\infty \frac{|f(\omega)|^2}{z - \omega}\,\mathrm{d}\omega\right]|1\rangle\ . \tag{136}$$

Consequently,

$$\begin{aligned} \langle 1|\frac{1}{z - H}|1\rangle &= \langle 1|P\,\frac{1}{z - H}\,P|1\rangle = \langle 1|[G(z)]^{-1}|1\rangle \\ &= \left[z - \omega_0 - \lambda^2\int_0^\infty \frac{|f(\omega)|^2}{z - \omega}\,\mathrm{d}\omega\right]^{-1}\ , \end{aligned} \tag{137}$$

which proves Theorem F.3. □

Now, let us assume that the function

$$h(\omega) = |f(\omega)|^2$$

is entire analytic on the complex variable ω. Then, the following result holds [51,47,66]:

Theorem F.4. *The function*,

$$\eta(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega}\,\mathrm{d}\omega\ , \tag{138}$$

is a complex analytic function with no singularities on the complex plane other than a branch cut coinciding with the positive semi-axis $\mathbb{R}^+$ *provided that* $\eta(0) > 0$. *It admits analytic continuations through the cut* $\eta_+(z)$, *from above to below* $\eta_-(z)$ *from below to above. The continuation* $\eta_+(z)$ *has a zero at* z_0 *with* $\mathrm{Im}\{z_0\} < 0$, *which is an analytic function on the coupling parameter* λ *on a neighborhood of zero. Analogously*, $\eta_-(z)$ *has a zero at* z_0^*, *which is also an analytic function of* λ.

The proof of this result is very technical and we refer the interested reader to the original sources [51,47,66]. The form of $\eta_\pm(z)$ is

$$\eta_\pm(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega \pm \mathrm{i}0}\,\mathrm{d}\omega\ , \tag{139}$$

where the signs plus and minus on the denominator of (139) indicate that the analytic function represented by the integral is an analytic continuation from above to below, $+$, or from below to above,$-$. Thus, z_0 and z_0^* are, respectively, the zeroes of the equations:

$$\eta_+(z) = 0 \quad \text{and} \quad \eta_-(z) = 0\ . \tag{140}$$

Theorem F.4 is also valid for other types of functions $h(\omega)$. See [5].

3.5.1. The Friedrichs model in RHS

The notation for the Friedrichs model presented so far is not, in our opinion, the most practical for explicit calculations. We shall use a new notation, in the spirit of the Dirac notation, for which rigged Hilbert spaces (RHS) are particularly useful. According to the Gelfand–Maurin spectral theorem, there exists a RHS $\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^\times$ such that $H_0\boldsymbol{\Phi} \subset \boldsymbol{\Phi}$, H_0 is continuous on $\boldsymbol{\Phi}$ and for any $\omega \in \mathbb{R}^+$, the absolutely continuous spectrum of H_0, there exists a $|\omega\rangle \in \boldsymbol{\Phi}^\times$ with $H_0|\omega\rangle = \omega|\omega\rangle$. As H_0 has the eigenvector $|1\rangle$, the spectral decomposition of H_0 is then

$$H_0 = \omega_0|1\rangle\langle 1| + \int_0^\infty \omega|\omega\rangle\langle\omega|\,\mathrm{d}\omega\ . \tag{141}$$

Then according to (118), V must have the following form:

$$V = \int_0^\infty [f^*(\omega)|\omega\rangle\langle 1| + f(\omega)|1\rangle\langle\omega|]\,\mathrm{d}\omega\ . \tag{142}$$

Also, the vector ψ in (112) can be written as

$$\psi = \alpha|1\rangle + \int_0^\infty \varphi(\omega)\,|\omega\rangle\,\mathrm{d}\omega\ . \tag{143}$$

Note that the set $\{|1\rangle, |\omega\rangle\}$, $\omega \in \mathbb{R}^+$, of generalized eigenvectors of H_0, forms a complete set in the sense that each $\psi \in \boldsymbol{\Phi}$ can be written as a superposition of these vectors. Thus, each vector in $\boldsymbol{\Phi}$ has

a component on the bound state $|1\rangle$ of H_0 and an infinite number of components, in the continuous spectrum of H_0.

With this notation, the action of V on ψ can be obtained if we note that

$$\langle 1|1\rangle = 1 \ , \tag{144}$$

$$\langle 1|\omega\rangle = \langle \omega|1\rangle = 0 \ , \tag{145}$$

$$\langle \omega|\omega'\rangle = \langle \omega'|\omega\rangle = \delta(\omega - \omega') \ . \tag{146}$$

To show (144), we observe that this is just the scalar product of vector (116) by itself. To show (146), let us consider the following scalar product in the old notation:

$$\left\langle \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix} \middle| \begin{pmatrix} 0 \\ \eta(\omega) \end{pmatrix} \right\rangle = \int_0^\infty \varphi^*(\omega)\eta(\omega)\,\mathrm{d}\omega \ . \tag{147}$$

In the new notation, this scalar product should be written as

$$\left\langle \int_0^\infty \varphi^*(\omega)\,\langle\omega|\mathrm{d}\omega \middle| \int_0^\infty \eta(\omega')\,|\omega'\rangle\,\mathrm{d}\omega' \right\rangle = \int_0^\infty \int_0^\infty \varphi^*(\omega)\eta(\omega')\langle\omega|\omega'\rangle\,\mathrm{d}\omega\,\mathrm{d}\omega' \ . \tag{148}$$

Should Eqs. (147) and (148) coincide, then Eq. (146) must follow. Observe that the Dirac delta is referred to the integration from 0 to ∞. Finally (145) follows from the fact that the states $|\omega\rangle$ span the lower component in (116) and functions in this component are orthogonal to $|1\rangle$.

Then, if we apply (142) to (143), we have

$$\begin{aligned} V\psi &= \int_0^\infty f^*(\omega)|\omega\rangle\,\mathrm{d}\omega\langle 1|\alpha|1\rangle + \int_0^\infty \int_0^\infty f(\omega)\,\varphi(\omega')|1\rangle\langle\omega|\omega'\rangle\,\mathrm{d}\omega\,\mathrm{d}\omega' \\ &= \alpha\int_0^\infty f^*(\omega)|\omega\rangle\,\mathrm{d}\omega + |1\rangle\int_0^\infty f(\omega)\varphi(\omega)\,\mathrm{d}\omega \ . \end{aligned} \tag{149}$$

This result coincides with (118). This shows how to operate in the new notation, which is more familiar to physicists.

3.6. The Gamow vectors for the Friedrichs model

Here, we intend to get the explicit form of the Gamow vectors for the Friedrichs model. We make the following assumptions:

(i) The function $\eta(z)$ has no singularities in the complex plane other than a branch cut coinciding with the positive semiaxis, i.e., the continuous spectrum of H.

(ii) The function $\eta(z)$ can be analytically continued through the cut. These are the functions $\eta_+(z)$ and $\eta_-(z)$ of the previous section. These extensions have poles located at the points z_0 and z_0^* as stated before.

(iii) Although resonance poles may, in principle, have arbitrary multiplicity, since in resonant scattering this is *not forbidden* by causality conditions [90], we shall assume first that our resonance poles are simple. In the next subsection, we shall obtain the Gamow vectors corresponding to a double pole resonance.

Thus, we intend to obtain the decaying Gamow vector $|f_0\rangle \in \boldsymbol{\Phi}_+^\times$ and the growing Gamow vector $|\tilde{f}_0\rangle \in \boldsymbol{\Phi}_-^\times$. In order to accomplish our goal, we do the following [13,8]: Let x be an arbitrary positive number ($x > 0$) and write the eigenvalue equation

$$(H - x)\,\Psi(x) = 0\ . \tag{150}$$

Since x belongs to the continuous spectrum of H, $\Psi(x)$ cannot be a normalized eigenvector of H and therefore, it should belong to either of the duals $\boldsymbol{\Phi}_\pm^\times$. As $|1\rangle$ and $|\omega\rangle$ form a complete system, we must have:

$$\Psi(x) = \alpha(x)|1\rangle + \int_0^\infty \psi(x,\omega)|\omega\rangle\,\mathrm{d}\omega\ . \tag{151}$$

If we apply H to (151), we obtain the following system of equations:

$$(\omega_0 - x)\alpha(\omega) + \lambda\int_0^\infty \psi(x,\omega)f^*(\omega)\,\mathrm{d}\omega = 0\ , \tag{152}$$

$$(\omega - x)\psi(x,\omega) + \lambda f(\omega)\alpha(\omega) = 0\ . \tag{153}$$

To solve this system, we write $\alpha(\omega)$ in terms of $\psi(x,\omega)$ using (153) and carry the result to (152). We obtain an integral equation, which gives one solution of the form

$$\Psi_+(x) = |x\rangle + \lambda f^*(x)\,\frac{1}{\eta_+(x)}\left\{|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{x - \omega + \mathrm{i}0}|\omega\rangle\,\mathrm{d}\omega\right\}\ . \tag{154}$$

This is a functional in $\boldsymbol{\Phi}_+^\times$. When applied to a vector in $\boldsymbol{\Phi}_+$, it gives an analytic function on the lower half-plane. We say that $\Psi_+(x)$ admits analytic continuation to the lower half-plane, in a weak sense. This continuation has a simple pole at z_0 so that we can write on a neighborhood of z_0:

$$\Psi_+(z) = \frac{C}{z - z_0} + o(z)\ . \tag{155}$$

From (150) and (155), we get

$$0 = (H - z)\Psi_+(z) = \frac{1}{z - z_0}(H - z)C + (H - z)o(z)\ , \tag{156}$$

which gives

$$(H - z_0)C = 0 \Rightarrow HC = z_0 C\ . \tag{157}$$

Therefore, the residue C of $\Psi(z)$ at the pole z_0 coincides, save for an irrelevant constant, with the decaying Gamow vector $|f_0\rangle$. To calculate its explicit form, note that (155) on a neighborhood of z_0 has the form

$$\Psi_+(z) \approx \frac{\text{constant}}{(z - z_0)}\left\{|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z - \omega + \mathrm{i}0}|\omega\rangle\,\mathrm{d}\omega\right\} + \mathrm{RT}\ , \tag{158}$$

where RT stand for "regular terms". Now, let us use the Taylor theorem to have

$$\frac{1}{z - \omega + \mathrm{i}0} = \frac{1}{z_0 - \omega + \mathrm{i}0} - \frac{z - z_0}{(z_0 - \omega + \mathrm{i}0)^2} + o(z)\ . \tag{159}$$

By replacing (159) in (158), we get

$$\Psi_+(z) \approx \frac{\text{constant}}{(z-z_0)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_0 - \omega + \mathrm{i}0} |\omega\rangle \, \mathrm{d}\omega \right\} + \mathrm{RT} \ . \tag{160}$$

Therefore, up to an irrelevant constant, we conclude that

$$C = |f_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_0 - \omega + \mathrm{i}0} |\omega\rangle \, \mathrm{d}\omega \ . \tag{161}$$

The system given by Eqs. (152) and (153) has another solution that can be analytically continued in the upper half-plane. This solution gives, using the same technique, the growing Gamow vector $|\tilde{f}_0\rangle$:

$$|\tilde{f}_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda f^*(\omega)}{z_0^* - \omega - \mathrm{i}0} |\omega\rangle \, \mathrm{d}\omega \ . \tag{162}$$

Note that the Gamow vectors depend on the coupling constant λ.

3.7. Double pole resonances

The zeroes of $\eta_\pm(z)$ at z_0 and z_0^* can be multiple, and this means that $\eta_+(z) = (z - z_0)^n G(z)$ with $G(z) \neq 0$ on a neighborhood of z_0. The question that arises now is whether these zeroes exist. We want to describe a typical situation, for which the zeroes at z_0 and at z_0^* are double and obtain the Gamow vectors.

Let us choose in (142) a form factor $f(\omega)$ such that [5]:

$$|f(\omega)|^2 = \frac{\sqrt{\omega}}{P(\omega)} \tag{163}$$

with

$$P(\omega) = (\omega - \alpha)(\omega - \alpha^*) \ . \tag{164}$$

Identities (163) and (164) determine the form factor $f(\omega)$ up to a phase. We have

$$\eta(z) = z - \omega_0 - \pi\lambda^2 \left\{ \frac{\sqrt{-z}}{P(z)} - \frac{1}{\alpha - \alpha^*} \left(\frac{\sqrt{-\alpha}}{z - \alpha} - \frac{\sqrt{-\alpha^*}}{z - \alpha^*} \right) \right\} \ . \tag{165}$$

If we make the change of variables:

$$z = p^2, \quad \alpha = b^2 \tag{166}$$

and write $\varphi(p) = \eta(p^2) = \eta(z)$, we obtain

$$\varphi(p) = p^2 - \omega_0 + \frac{\mathrm{i}\pi\lambda^2}{(b - b^*)(p + b)(p - b^*)} \ . \tag{167}$$

Then, if $\eta_+(z)$ has a double zero at z_0, $\varphi(p)$ has a double zero at p_0 with $z_0 = p_R^2$ [5]. The condition that $\varphi(p)$ has a double pole at p_0 reads

$$\varphi(p_0) = 0, \quad \varphi'(p_0) = 0, \quad \varphi''(p_0) \neq 0 \ . \tag{168}$$

The two complex equations (168) give a system of four real equations for six parameters, which are: ω_0, λ, Re p_0, Im p_0, Re b and Im b. We have two free parameters that, we choose to be ω_0 and λ. We can write the other four parameters in terms of these two and obtain:

$$b = \omega_0^{1/2} + 2\mathrm{i}\left(\frac{\pi\lambda^2}{16\omega_0}\right)^{1/3} \tag{169}$$

and

$$p_0 = \left[\omega_0 - \left(\frac{\pi\lambda^2}{16\omega_0}\right)^{2/3}\right] - \left(\frac{\pi\lambda^2}{16\omega_0}\right)^{1/3} \quad . \tag{170}$$

However, the parameters ω_0 and λ are not completely independent if we impose the condition Im $p_0 < 0$, since this implies that

$$\omega_0 - \left(\frac{\pi\lambda^2}{16\omega_0}\right)^{2/3} \geqslant 0 \ . \tag{171}$$

The condition Im $p_0 < 0$ comes from $\eta(0) > 0$ and we impose the latter in order to restrict the singularities of $\eta(z)$ to those on the branch cut [47,66]. The resonance poles are singularities of the *analytic continuations* of $\eta(z)$, or for $\varphi(p)$ on the upper half-plane [66].

If we carry (169) and (170) to (167), we obtain

$$\varphi(p) = -\frac{(p-p_0)^2(p+p_0^*)^2}{(p+t)(p-t^*)} \ , \tag{172}$$

where t is a complex number different from p_0 or p_0^*. We observe that $\varphi(p)$ has double zeroes at p_R and p_R^* and therefore, so has $\eta(z)$ at z_0 and z_0^*.

Due to the fact that the analytic continuation of (154) exists on a neighborhood of z_0, we can now write

$$\Psi_+(z) \approx \frac{C_1}{(z-z_0)^2} + \frac{C_2}{z-z_0} + \mathrm{RT} \tag{173}$$

and

$$\begin{aligned}\Psi_+(z) &\approx \frac{\text{constant}}{(z-z_0)^2}\left\{|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z-\omega+\mathrm{i}0}\,|\omega\rangle\,\mathrm{d}\omega\right\} \\ &= \frac{\text{constant}}{(z-z_0)^2}\left\{|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z_0-\omega+\mathrm{i}0}|\omega\rangle\,\mathrm{d}\omega\right. \\ &\quad \left. -\lambda(z-z_0)\int_0^\infty \frac{f(\omega)}{(z_0-\omega+\mathrm{i}0)^2}\,|\omega\rangle\,\mathrm{d}\omega\right\} \ .\end{aligned} \tag{174}$$

Comparing (174) with (173), we obtain

$$C_1 = |1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z_0-\omega+\mathrm{i}0}\,|\omega\rangle\,\mathrm{d}\omega \ , \tag{175}$$

$$C_2 = -\lambda\int_0^\infty \frac{f(\omega)}{(z_0-\omega+\mathrm{i}0)^2}\,|\omega\rangle\,\mathrm{d}\omega \ . \tag{176}$$

We must show that C_1 and C_2 are indeed the Gamow vectors we are looking for. More precisely

$$|f_0\rangle = C_1, \quad |f_1\rangle = C_2 \ . \tag{177}$$

Due to the properties of $|f_0\rangle$ and $|f_1\rangle$ presented in Sections 3.2 and 3.3, to get (177) we have to show that

$$HC_1 = z_R C_1, \quad HC_2 = z_R C_2 + C_1 \ . \tag{178}$$

From (175), there is no doubt that $C_1 = |f_0\rangle$. To obtain the second relation in (178), we apply H to (176) and get

$$\begin{aligned} HC_2 &= -\lambda \int_0^\infty d\omega' \omega' |\omega'\rangle\langle\omega'| \int_0^\infty d\omega \, \frac{f(\omega)}{(z_R - \omega + i0)^2} \, |\omega\rangle \\ &\quad -\lambda^2 \int_0^\infty d\omega' f^*(\omega) |1\rangle\langle\omega'| \lambda \int_0^\infty d\omega \, \frac{f(\omega)}{(z_R - \omega + i0)^2} \, |\omega\rangle \\ &= -\lambda \int_0^\infty d\omega \, \frac{\omega f(\omega)}{(z_R - \omega + i0)^2} \, |\omega\rangle - \left[\lambda^2 \int_0^\infty d\omega \, \frac{|f(\omega)|^2}{(z_R - \omega + i0)^2}\right] |1\rangle \ . \end{aligned} \tag{179}$$

The coefficient of $|1\rangle$ in (179) is equal to one. To show it, let us go back to (139) and derive it with respect to z. We get

$$\eta'_+(z) = -1 - \lambda^2 \int_0^\infty d\omega \, \frac{|f(\omega)|^2}{(z - \omega + i0)^2} \ . \tag{180}$$

Since $\varphi'(p_0) = 0$ implies $\eta'(z_0) = 0$ [5], we have that

$$\eta'_+(z_0) = 0 = -1 - \lambda^2 \int_0^\infty d\omega \, \frac{|f(\omega)|^2}{(z_0 - \omega + i0)^2} \ , \tag{181}$$

which supports our claim. After (181) and (179) can be written as

$$\begin{aligned} HC_2 &= |1\rangle + \lambda \int_0^\infty d\omega \, \frac{(-\omega + z_R) f(\omega)}{(z_0 - \omega + i0)^2} \, |\omega\rangle \\ &\quad + z_0 \left[-\lambda \int_0^\infty d\omega \, \frac{f(\omega)}{(z_0 - \omega + i0)^2} \, |\omega\rangle\right] \\ &= |1\rangle + \lambda \int_0^\infty d\omega \, \frac{f(\omega)}{z_0 - \omega + i0} \, |\omega\rangle + z_0 \left[-\lambda \int_0^\infty d\omega \, \frac{f(\omega)}{(z_0 - \omega + i0)^2} |\omega\rangle\right] \\ &= C_1 + z_0 C_2 \ , \end{aligned} \tag{182}$$

which shows that $C_2 = |f_1\rangle$.

On the subspace spanned by $|f_0\rangle$ and $|f_1\rangle$, the total Hamiltonian H exhibits a block diagonal form:

$$H = \begin{pmatrix} z_0 & 1 \\ 0 & z_0 \end{pmatrix} \ . \tag{183}$$

Analogously, one obtains for $|\tilde{f}_0\rangle$ and $|\tilde{f}_1\rangle$ the following expressions:

$$|\tilde{f}_0\rangle = |1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \, \frac{f(\omega)}{z_0^* - \omega - \mathrm{i}0} \, |\omega\rangle \ , \tag{184}$$

$$|\tilde{f}_1\rangle = -\lambda \int_0^\infty \mathrm{d}\omega \, \frac{f(\omega)}{(z_0^* - \omega - \mathrm{i}0)^2} \, |\omega\rangle \ . \tag{185}$$

Remark. Another model producing a double pole resonance has been discussed by Mondragón and coworkers [86,87,10]. In their model, a double pole resonance is produced in one-dimensional scattering with a double delta barrier. The potential has the form

$$V = \frac{\pi}{\alpha} \, \delta(x-a) + \frac{\pi}{\beta} \, \delta(x-b) \ ,$$

where a, b, α and β are real parameters, and it gives an infinite number of resonances such that pairs of them coincide for certain values of the parameters [86,87,10].

3.8. The choice of the space $\boldsymbol{\Phi}$

There are several possibilities to choose the space $\boldsymbol{\Phi}$, but we request that $V\boldsymbol{\Phi} \subset \boldsymbol{\Phi}$ and that V be continuous on $\boldsymbol{\Phi}$, so that H is also continuous. This depends on the form factor function $f(\omega)$. If we choose as in [13]

$$\boldsymbol{\Phi}_\pm = \mathbb{C} \oplus S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+} \ , \tag{186}$$

where the spaces $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ (see Section 2), we need that $f^*(\omega) \in S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$. In this case, both H_0 and H have the following properties:

(i) The spaces $\boldsymbol{\Phi}_\pm$ are reduced by both H_0 and H. This means that $H_0\boldsymbol{\Phi}_\pm \subset \boldsymbol{\Phi}_\pm$ and $H\boldsymbol{\Phi}_\pm \subset \boldsymbol{\Phi}_\pm$.

(ii) Both Hamiltonians H_0 and H are continuous on $\boldsymbol{\Phi}_\pm$.

(iii) Using the duality formula

$$\langle A\,\varphi | F\rangle = \langle \varphi | AF\rangle, \quad \varphi \in \boldsymbol{\Phi}, \quad F \in \boldsymbol{\Phi}^\times \ ,$$

where A is either H_0 or H, we can extend both operators to the duals $\boldsymbol{\Phi}^\times_\pm$. Then, it is possible to show that these extensions are continuous operators on $\boldsymbol{\Phi}^\times_\pm$, when we endow $\boldsymbol{\Phi}^\times_\pm$ with the weak topology (see [105]). The condition $f^*(\omega) \in S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ is however too restrictive (see [5]). For practical purposes, the function $f(\omega)$ should be a real function of the real variable ω admitting analytic continuation to the whole complex plane with possibly a branch cut along the positive semiaxis. In this case, the property $H\boldsymbol{\Phi}_\pm \subset \boldsymbol{\Phi}_\pm$ is false in general. There is, however, a way out as H can be shown to be a continuous operator from $\boldsymbol{\Phi}^\times_\pm$ into $\boldsymbol{\Phi}^\times_\pm$, for many choices of $f(\omega)$. Then, we assume in general that

$$H\boldsymbol{\Phi}^\times_\pm \subset \boldsymbol{\Phi}^\times_\pm \ . \tag{187}$$

The possibility of defining Gamow vectors for the Friedrichs model, results from this idea. In fact, for a wide class of form factors $f(\omega)$, it is possible to show that the Møller wave operators $\boldsymbol{\Omega}_{\mathrm{OUT}}$ and $\boldsymbol{\Omega}_{\mathrm{IN}}$ exist for the pair $\{H_0, H\}$ and are asymptotically complete [47]. In this case, the poles of

the analytical continuation of the S-matrix in the energy representation, $S(E)$, through the spectrum of H, coincide with the poles of the analytic continuations of the reduced resolvent through the cut [15]. These poles coincide with the zeroes of the functions $\eta_{\pm}(z)$.

These poles appear in complex conjugate pairs. In the simplest case, we have only two and are simple: $z_0 = E_R - i\Gamma/2$ and $z_0^* = E_R + i\Gamma/2$ with $E_R > 0$, $\Gamma > 0$. Since z_0 and z_0^* are the poles of the analytic continuation of $S(E)$, Gamow vectors can be defined as the eigenvectors of H with respective eigenvalues given by z_0 and z_0^*. Since H is Hermitian, Gamow vectors cannot belong to $\mathscr{H}$.

On the other hand, H can be extended into $\boldsymbol{\Phi}_{\pm}^{\times}$. We can show that

(i) There is an eigenvector of H in $\boldsymbol{\Phi}_{+}^{\times}$ with eigenvalue z_0 ($H|f_0\rangle = z_0|f_0\rangle$) and an eigenvector of H in $\boldsymbol{\Phi}_{-}^{\times}$ with eigenvalue z_0^* ($H|\tilde{f}_0\rangle = z_0^*|\tilde{f}_0\rangle$).

(ii) Time evolution for $|f_0\rangle$ and $|\tilde{f}_0\rangle$ is given by (74) and (75), respectively.

Therefore, the vectors $|f_0\rangle$ and $|\tilde{f}_0\rangle$ should be the decaying and growing Gamow vectors, respectively. We shall show how to obtain them in the next subsection. We shall also study the possibility that the resonance poles are double and give a situation in which it is realized.

3.9. Gamow states as continuous linear functionals over analytical test functions

In this subsection, we review a different approach based in the use of tempered ultra-distributions [33,34]. Tempered ultra-distributions are functionals on a test vector space, here called ξ, that we describe as follows:

As a vector space, the test space ξ is the space of entire analytic functions such that, when considered as functions defined on the real axis, are Schwartz functions. In other words, if $\hat{\phi}(z) \in \xi$ with $z = x + iy$, then

$$\hat{\phi}(z)|_{y=0} = \hat{\phi}(x) \in S \ .$$

The space ξ is rather big as it contains the Fourier transforms of the functions on S which vanish outside a bound interval [104]. This space is dense in the space of square integrable functions on the whole real line, $L^2(\mathbb{R})$, and therefore, so is ξ. Thus, $\xi \subset L^2(\mathbb{R})$ and ξ is densely defined on $L^2(\mathbb{R})$.

Let us consider on ξ the following set of norms:

$$\|\hat{\phi}\|_n = \sup_{|z|=n} |\hat{\phi}(z)|, \quad n = 1, 2, 3 \ldots \ . \tag{188}$$

The scalar product on ξ

$$\langle \hat{\psi}|\hat{\phi}\rangle = \int_{-\infty}^{+\infty} dE \hat{\psi}^*(E)\hat{\phi}(E) \ , \tag{189}$$

provides the following norm:

$$\|\hat{\phi}\|^2 = \langle \hat{\phi}|\hat{\phi}\rangle \ . \tag{190}$$

The set of norms (188) along the norm (190) endow ξ with a metric topology, with the following properties:

(i) Since ξ is a subspace of $L^2(\mathbb{R})$ and it possesses the norm of $L^2(\mathbb{R})$, it is clear that the canonical injection

$$i : \xi \mapsto L^2(\mathbb{R}), \quad i[\hat{\phi}(z)] = \hat{\phi}(z), \quad \forall \hat{\phi}(z) \in \xi$$

is continuous.

(ii) Since ξ has the Hilbert space norm of $L^2(\mathbb{R})$, it then results that any Cauchy sequence in ξ is also a Cauchy sequence in $L^2(\mathbb{R})$. The converse is, however, not true as we have on ξ an infinite number of nonequivalent norms. Thus, the topology on ξ is *finer* than the Hilbert space topology that ξ has received from $L^2(\mathbb{R})$. This means in particular that ξ has less Cauchy sequences and more open sets than $L^2(\mathbb{R})$.

(iii) The topology on ξ has the property of nuclearity, exactly as S. We do not want to go further on this comment, because nuclearity is a very technical mathematical tool.

The space of *linear* continuous functionals on ξ is the space ξ' of tempered ultra-distributions. It fulfills the following important property:

$$\xi \subset L^2(\mathbb{R}) \subset \xi' \ . \tag{191}$$

Thus, (191) is a RHS (although the functionals on ξ' are linear and not anti-linear). We should remark that, according to the Gelfand Maurin theorem, any hermitian operator A on ξ admitting one and only one self-adjoint extension on $L^2(\mathbb{R})$ (hermitian operators with this property are called essentially self-adjoint operators) has a complete system of eigenvectors in ξ' whose respective eigenvalues are in the Hilbert space spectrum of A.

Related to the RHS in (191), we have another RHS which is obtained with the use of the inverse Fourier transform on (191). For any $\hat{\phi}(E) \in \xi$, we have

$$\phi(t) = \mathscr{F}^{-1}\{\hat{\phi}(E)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}Et} \hat{\phi}(E)\, \mathrm{d}E \ . \tag{192}$$

Since the inverse Fourier transform of a Schwartz function is also a Schwartz function, for each $\hat{\phi}(E)$, the $\phi(t)$ in (192) is a Schwartz function. We shall call $\tilde{\xi}$ the vector space of functions of the form (192). Thus,

$$\tilde{\xi} = \mathscr{F}^{-1}\xi \ .$$

The inverse Fourier transform $\mathscr{F}^{-1}$ is an one-to-one mapping from ξ onto $\tilde{\xi}$. We can endow $\tilde{\xi}$ with the topology transported from ξ into $\tilde{\xi}$ by $\mathscr{F}^{-1}$. This means that if $\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_n, \ldots$ is a Cauchy sequence in ξ, then, $\phi_1 = \mathscr{F}^{-1}\hat{\psi}_1, \phi_2 = \mathscr{F}^{-1}\hat{\psi}_2, \phi_3 = \mathscr{F}^{-1}\hat{\psi}_3, \ldots$ is a Cauchy sequence in $\tilde{\xi}$. The space ξ is also dense in $L^2(\mathbb{R})$ and, if $\tilde{\xi}'$ is the space of all linear continuous functionals on $\tilde{\xi}$, then,

$$\tilde{\xi} \subset L^2(\mathbb{R}) \subset \tilde{\xi}'$$

is a RHS. We can extend $\mathscr{F}^{-1}$ as a mapping from ξ' onto $\tilde{\xi}'$ by means of the duality formula:

$$\langle \phi | F \rangle = \langle \mathscr{F}^{-1}\hat{\phi} | \mathscr{F}^{-1}\hat{F} \rangle = \langle \hat{\phi} | \hat{F} \rangle \ .$$

At this point, the connection between this formalism and the formalism presented in Section 3.1 is established. In fact, schematically,

$$\text{RHS} \leftrightarrow \xi \subset L^2(\mathbb{R}) \subset \xi' \ ,$$

$$\text{RHS} \leftrightarrow \tilde{\xi} \subset L^2(\mathbb{R}) \subset \tilde{\xi}' \ ,$$

$$\text{RHS} \leftrightarrow \boldsymbol{\Phi}_{\pm} \subset \mathscr{H} \subset \boldsymbol{\Phi}_{\pm}^{\times} \ .$$

Remark. The relation between S, S' (space of linear continuous functionals on S also called space of tempered distributions), ξ, ξ', $\tilde{\xi}$ and $\tilde{\xi}'$ is given by:

$$\xi \subset S \subset L^2(\mathbb{R}) \subset S' \subset \xi'$$

and

$$\tilde{\xi} \subset S \subset L^2(\mathbb{R}) \subset S' \subset \tilde{\xi}' \ .$$

Note that if $S(x) \in S'$, the Dirac formula gives

$$S(x) = \int_{-\infty}^{\infty} S(y)\, \delta(x-y)\, \mathrm{d}y \ .$$

Analogously, we can obtain the value of $\hat{\psi}_{\mathrm{c}}(z) \in \xi'$ at any complex value z (with $\mathrm{Im}\, z \neq 0$), if we note that the ultra-distributions in ξ' can be represented as complex analytic functions with a branch cut on a real interval. Therefore, the desired formula is [107]

$$\hat{\psi}_{\mathrm{c}}(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{1}{E-z}\, \hat{\psi}(E)\, \mathrm{d}E \ , \tag{193}$$

where

$$\hat{\psi}(E) = \hat{\psi}_{\mathrm{c}}(E + \mathrm{i}0) - \hat{\psi}_{\mathrm{c}}(E - \mathrm{i}0) \ . \tag{194}$$

and $\hat{\psi}_{\mathrm{c}}(E + \mathrm{i}0)$ and $\hat{\psi}_{\mathrm{c}}(E - \mathrm{i}0)$ are, respectively, the boundary values from above to below and from below to above of the complex analytic function $\hat{\psi}_{\mathrm{c}}(z)$ (to illustrate the point see Eqs. (39) and (193)).

Now, let H be the total hamiltonian and assume that it has a simple continuous spectrum from E_0 to E_1 (in general H is semibound so that $E_1 = \infty$). Due to the above-mentioned Gelfand–Maurin theorem, there exists functionals $|E\rangle \in \xi'$ such that $H|E\rangle = E|E\rangle$ for (almost in the sense of the Lebesgue measure) all $E \in (E_0, E_1)$.

Now, let $z_{\mathrm{R}} = E_{\mathrm{R}} - \mathrm{i}\Gamma/2$ and $z_{\mathrm{R}}^* = E_{\mathrm{R}} + \mathrm{i}\Gamma/2$ its complex conjugate just as in previous sections. Let us take those $\psi_{\mathrm{c}}(z) \in \xi'$ such that the difference (194) vanishes outside the interval (E_0, E_1). In this case, formula (193) can be written as

$$\hat{\psi}_{\mathrm{c}}(z_{\mathrm{R}}^*) = \frac{1}{2\pi \mathrm{i}} \int_{E_0}^{E_1} \frac{1}{E - z_{\mathrm{R}}^*}\, \hat{\psi}(E)\, \mathrm{d}E \ , \tag{195}$$

and also as

$$[\hat{\psi}_{\mathrm{c}}(z_{\mathrm{R}})]^* = \frac{1}{2\pi \mathrm{i}} \int_{E_0}^{E_1} \frac{1}{z_{\mathrm{R}} - E}\, [\hat{\psi}(E)]^*\, \mathrm{d}E \ . \tag{196}$$

It is customary to write the difference $\hat{\psi}(E)$ (see (194)) in the Dirac notation as $\hat{\psi}(E) = \langle E|\hat{\psi}\rangle$. Then from (195), we have

$$\hat{\psi}(z_R^*) = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{E - z_R^*}\langle E|\psi\rangle \, dE \tag{197}$$

and using the convention $\langle\psi|E\rangle^* = \langle E|\psi\rangle$, we have from (196):

$$[\hat{\psi}(z_R)]^* = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{z_R - E}\langle\psi|E\rangle \, dE \ . \tag{198}$$

We now define

$$\langle \tilde{f}_0| = \langle z_R^*| = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{E - z_R^*}\langle E| \, dE \tag{199}$$

and

$$|f_0\rangle = |z_R\rangle = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{z_R - E}|E\rangle \, dE \ . \tag{200}$$

In consequence:

$$\hat{\psi}_c(z_R^*) = \langle \tilde{f}_0|\psi\rangle, \quad [\hat{\psi}(z_R)]^* = \langle\psi|f_0\rangle \ . \tag{201}$$

Compare the form of these functionals to (67).

Vectors (199) and (200) are, respectively, the left and right Gamow vectors in this presentation. For $n = 0, 1, 2, \ldots$, also note that

$$\begin{aligned} H^n|f_0\rangle &= \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{z_R - E} H^n|E\rangle \, dE \\ &= \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{z_R - E} E^n|E\rangle \, dE \ . \end{aligned} \tag{202}$$

If $\psi(z)$ is an arbitrary function in ξ, we have

$$\langle\psi|H^n|f_0\rangle = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{E^n}{z_R - E}\langle\psi|E\rangle \, dE = z_R^n \langle\psi|f_0\rangle \ , \tag{203}$$

so that if we omit the arbitrary function $\psi(z)$, we have

$$H^n|f_0\rangle = z_R^n|f_0\rangle, \quad n = 0, 1, 2, \ldots \ . \tag{204}$$

Analogously, for $n = 0, 1, 2, \ldots$, we have

$$\langle \tilde{f}_0 | H^n = z_R^n \langle \tilde{f}_0 | \ . \tag{205}$$

The time evolution is given by

$$\langle \psi | e^{-iHt} | f_0 \rangle = e^{-iz_R} [\hat{\psi}(z_R^*)]^* = e^{-itE_R} e^{-\Gamma t/2} \langle \psi | f_0 \rangle \ . \tag{206}$$

Of course, if we omit the arbitrary $\psi(z) \in \xi$, we have:

$$e^{-iHt} | f_0 \rangle = e^{-itE_R} e^{-\Gamma t/2} | f_0 \rangle \ . \tag{207}$$

This exponential evolution is valid for all values of time and therefore it does not lead to semi-group time evolution. Consequently, this description based on ultra-distributions is not suitable for a description of quantum irreversibility, in the sense of Prigogine [13,96,97], as it is the description based on Hardy functions.

4. Observables

Generally speaking, by quantum mechanics we mean the theory and its interpretation. Central to this point is the assignment of mean values to observables acting on states. So far, we have not addressed this problem in relation to Gamow states and we shall devote this section to present the state of the art. As it will become evident from our discussion, the notion that the use of Gamow states violates the quantum mechanical probabilistic interpretation is wrong. As we said in the Introduction, and as it has been shown in the previous section, probability considerations are avoided in RHS.

The question whether it is legitimate to define the mean value of an observable on a Gamow vector is still open.

According to the postulates of the elementary quantum mechanics, the definition of the expectation value of an observable A on a certain state ρ is given by $\langle A \rangle = \text{trace}\, \rho A$. It is desirable to generalize this definition to accommodate expectation values on Gamow states. This has been attempted to in Ref. [57]. As the reader may have already guessed the question of the physical interpretation of the expectation value of an observable on Gamow states seems to be in conflict with the standard, probabilistic, interpretation of quantum mechanics. It was clear, even when Gamow postulated the existence of resonant states, that notions like uncertainties, the conservation of probability, etc., exceeded the framework based on the Hilbert space formulation of quantum mechanics. In the following we shall concentrate on the various possibilities offered by the use and properties of Gamow vectors.

Let us briefly summarize some of the possibilities that finds in the literature [42,43], concerning the energy as the expectation value of the Hamiltonian on a resonance. They are:

1. The mean value of the energy of a decaying state must be zero, because the energy of a decaying process should be invariant. A non-zero energy will be in contradiction with the principle of conservation of the energy [88,39,38,36].

2. The energy average of a Gamow state should be complex because the Gamow state is an eigenvalue of H with complex energy.

3. Gamow states admit a representation as *normalized* vectors in a Hilbert space in which the Hamiltonian *is not* a self-adjoint operator and it has a spectrum of eigenvalues extending from

$-\infty$ to ∞. In this case, we can define scalar products of Gamow vectors and a mean value of the energy, which is real [20].

4. If we admit that Gamow states are genuine quantum states they must have a real energy average and it should be determined from first principles [20,17,18].

Next, let us discuss these four possibilities.

4.1. Vanishing average values

It was Nakanishi the first to proposed this idea [88]. In fact, if $H|f_0\rangle = z_R|f_0\rangle$ and $\langle f_0|H = z_R^*\langle f_0|$, we have that $\langle f_0|H|f_0\rangle = z_R\langle f_0|f_0\rangle = z_R^*\langle f_0|f_0\rangle$. This implies that $(z_R - z_R^*)\langle f_0|f_0\rangle = 0 \Rightarrow \langle f_0|f_0\rangle = 0$ and, therefore, $\langle f_0|H|f_0\rangle = 0$. The weak point of this argument is that the bracket $\langle f_0|f_0\rangle$ is not defined. Further attempts to define $\langle f_0|f_0\rangle$ have been made, but the results are not convincing from a mathematical point of view [36,96,91]. There is another possibility to rescue this idea and give to it a precise mathematical form. It is based on the definition of a Gamow state as an exponentially decaying functional over an algebra of observables [37]. The mean value of any observable on the Gamow functional is the action of the functional on the observable. Thus, we can define the trace of any functional, as the action of this functional on the identity, and the mean value of the energy on this functional, as its action on the Hamiltonian. The Gamow functional has zero trace and the mean value of its energy is equal to zero. Furthermore, the action of the Gamow functional on any power of the Hamiltonian is always zero. This suggest that the Gamow functional cannot represent a physical state. For details see [37].

4.2. Complex average values

In specific models, like Friedrichs's model, the bracket $\langle \tilde{f}_0|f_0\rangle$ is well defined and its value is one [13]. If we try to obtain this result in a general model independent setting we conclude that $\langle \tilde{f}_0|f_0\rangle$ can be defined as a distribution kernel and that it has the value one, although it is not clear if this is the unique choice [55]. If we now define $\Pi = |f_0\rangle\langle \tilde{f}_0|$, it is now obvious that $\Pi^2 = \Pi$. This property suggests that Π could be taken as the density operator for the decaying Gamow vector $|f_0\rangle$. Now if it would be possible to define $\mathrm{Tr}\{H\Pi\}$ and this would be a candidate for the average value of H on $|f_0\rangle$ [57]. In fact, with the help of some generalized spectral decompositions [55] for the Hamiltonian in terms of the Gamow vectors and the generalized eigenvectors of H with eigenvalues in the continuous spectrum of H, we can define this trace in such a way that $\mathrm{Tr}\{H\Pi\} = \langle \tilde{f}_0|H|f_0\rangle$ [55,57], thus

$$\langle \tilde{f}_0|H|f_0\rangle = z_R\langle \tilde{f}_0|f_0\rangle = z_R \ . \tag{208}$$

Yet this result is not acceptable from the physical point of view. Due to the uncertainty principle we cannot measure *simultaneously* the real part of z_R, which is the resonant energy, and its imaginary part, which is proportional to the inverse of the half life. Thus, z_R cannot be the *average* of any measurement process and cannot be accepted as the energy average.

Thus, the above generalization, which in principle seems to be the most natural definition of the mean value of the energy for a Gamow vector, becomes untenable.

Also, from these considerations, we conclude that the energy average of a Gamow vector, if it can be defined, should be real.

Actually, the same problem arises when dealing with the complex scaling formalism [101,108]. Gamow vectors are, in complex scaling, eigenvectors with complex eigenvalues of an analytically dilated non-self-adjoint Hamiltonian $H(\theta)$, where θ is a complex parameter such that $H(\theta=0)=H$ is the total, interacting, hamiltonian $H=H_0+V$. These complex eigenvalues, z_{R}, are independent of θ [114], and coincide with poles of the S-matrix in the energy representation, as it was shown in [4]. Therefore, they are truly resonance eigenvalues. However, the corresponding eigenvectors $|f_\theta\rangle$ are dependent on θ, as dictated by the eigenvalue equation

$$H(\theta)|f_\theta\rangle = z_{\mathrm{R}}\,|\,f_\theta\rangle\ . \tag{209}$$

The eigenvector $|f_\theta\rangle$ is a normalized vector on Hilbert space and, therefore, its norm can be unity. Then, a natural definition for the mean value of the energy on a Gamow state $|f_\theta\rangle$ is

$$\langle f_\theta|H(\theta)|f_\theta\rangle = z_{\mathrm{R}}\ . \tag{210}$$

Since $|f_\theta\rangle$ is normalized the above expectation value is well defined. Again, this definition gives the same untenable result, as we have discussed before.

4.3. Real average values: Bohm interpretation

This point of view is based in the idea that it is possible to construct a rigged Hilbert space (RHS), in which the Gamow vector is a vector in the Hilbert space, under the following conditions:

(i) The continuous spectrum of H is the whole real axis.

(ii) H is not self adjoint, although it is still symmetric, i.e., $\langle\phi|H\psi\rangle=\langle H\phi|\psi\rangle$ for all ϕ and ψ in the domain of H.

(iii) From the point of view of the Hilbert space, the Gamow vector is not in the domain of H, but the action of H on the Gamow vector is well defined in the dual space, that includes the Hilbert space.

To this end, let us consider the RHS spaces in the lower row in (71) and (72). These spaces are:

$$S\cap\mathscr{H}_-^2\subset\mathscr{H}_-^2\subset(S\cap\mathscr{H}_-^2)^\times \tag{211}$$

and

$$S\cap\mathscr{H}_+\subset\mathscr{H}_+\subset(S\cap\mathscr{H}_+)^\times\ . \tag{212}$$

Let us define now

$$\psi^{\mathrm{G}}=(\theta_-^{-1})^\times U\boldsymbol{\Omega}_{\mathrm{IN}}^{-1}|\tilde{f}_0\rangle \tag{213}$$

and

$$\psi^{\mathrm{D}}=(\theta_+^{-1})^\times U\boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}|f_0\rangle\ . \tag{214}$$

From (213) and (214) is not possible to obtain the explicit form of ψ^{G} and ψ^{D}. In order to get these objects, let us take arbitrary vectors $\psi_\pm\in\boldsymbol{\Phi}_\pm$ and consider the functions

$$\psi_-(E)=(\theta_-^{-1})\,U\boldsymbol{\Omega}_{\mathrm{IN}}^{-1}\psi_-\quad\text{and}\quad\psi_+(E)=(\theta_+^{-1})U\boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}\psi_+\ ,$$

and their complex conjugates:

$$\psi_-^{\#}(E)=[\psi_-(E)]^*\in S\cap\mathscr{H}_+^2\quad\text{and}\quad\psi_+^{\#}(E)=[\psi_+(E)]^*\in S\cap\mathscr{H}_-^2\ .$$

We recall that $\psi_\pm(E) \in S \cap \mathscr{H}^2_\pm$, and they are, therefore, defined on the whole real axis, $-\infty < E < \infty$. The duality formulas of Section 2 and the Titchmarsh theorem [110] give

$$\langle \psi_-(E)|\psi^{\rm G}\rangle = \langle \psi_-|\tilde{f}_0\rangle = -\frac{1}{2\pi {\rm i}} \int_{-\infty}^{\infty} \frac{\psi_-^{\#}(E)\,{\rm d}E}{E - z_{\rm R}^*} \tag{215}$$

and

$$\langle \psi_+(E)|\psi^{\rm D}\rangle = \langle \psi_+|f_0\rangle = \frac{1}{2\pi {\rm i}} \int_{-\infty}^{\infty} \frac{\psi_+^{\#}(E)\,{\rm d}E}{E - z_{\rm R}} \ . \tag{216}$$

Therefore,

$$\psi^{\rm G} = -\frac{1}{2\pi {\rm i}} \frac{1}{E - z_{\rm R}^*} \quad \text{and} \quad \psi^{\rm D} = \frac{1}{2\pi {\rm i}} \frac{1}{E - z_{\rm R}} \ . \tag{217}$$

Note that

$$\psi^{\rm G} \in (S \cap \mathscr{H}^2_-)^\times \quad \text{and} \quad \psi^{\rm D} \in (S \cap \mathscr{H}^2_+)^\times \ .$$

Since the operators $(\theta_-^{-1})^\times\, U\, \boldsymbol{\Omega}_{\rm IN}^{-1}$ and $(\theta_+^{-1})^\times U \boldsymbol{\Omega}_{\rm OUT}^{-1}$ are one to one and onto transformations between dual spaces, the functionals $\psi^{\rm G}$ and $\psi^{\rm D}$ represent Gamow vectors in $(S \cap \mathscr{H}^2_-)^\times$ and $(S \cap \mathscr{H}^2_+)^\times$, respectively.

We see that:

(i) These Gamow vectors are represented by square integrable functions in $L^2(\mathbb{R})$. Moreover, by using the definition of Hardy functions, it is easy to show that

$$\psi^{\rm G} \in \mathscr{H}^2_- \quad \text{and} \quad \psi^{\rm D} \in \mathscr{H}^2_+ \ . \tag{218}$$

(ii) The operator that represents the Hamiltonian on the RHS $S \cap \mathscr{H}^2_\pm \subset \mathscr{H}_\pm \subset (S \cap \mathscr{H}^2_\pm)^\times$ is given by

$$(\theta_+^{-1})\, U\, \boldsymbol{\Omega}_{\rm OUT}^{-1} H \boldsymbol{\Omega}_{\rm OUT} U^{-1} \theta_+ \ ,$$

on $S \cap \mathscr{H}^2_+$ and

$$(\theta_-^{-1}) U \boldsymbol{\Omega}_{\rm IN}^{-1} H \boldsymbol{\Omega}_{\rm IN} U^{-1} \theta_-$$

on $S \cap \mathscr{H}^2_-$. Observe that $\boldsymbol{\Omega}_{\rm OUT}^{-1} H \boldsymbol{\Omega}_{\rm OUT} = \boldsymbol{\Omega}_{\rm IN}^{-1} H \boldsymbol{\Omega}_{\rm IN} = H_0$. Since U diagonalizes H_0, we have that $U H_0 U^{-1} = \mathscr{E}$ is the multiplication operator on $S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+}$ ($\mathscr{E}\psi(E) = E\psi(E)$). Thus, $\theta_\pm^{-1} \mathscr{E}\ \theta_\pm$ are the multiplication operators on $S \cap \mathscr{H}^2_\pm$, respectively. These operators are the restrictions of the multiplication operator on $L^2(\mathbb{R})$, which we also denote by $\mathscr{E}$.

Now, let us observe that neither

$$\mathscr{E}\psi^{\rm G} = -(2\pi {\rm i})^{-1} \frac{E}{E - z_{\rm R}^*} \quad \text{nor} \quad \mathscr{E}\psi^{\rm D} = (2\pi {\rm i})^{-1} \frac{E}{E - z_{\rm R}} \ , \tag{219}$$

are square integrable functions on $L^2(\mathbb{R})$. This means that, although the Gamow vectors may be represented by square integrable functions, these functions are not in the domain of the operator $\hat{E}$ (i.e. the Hamiltonian).

(iii) Formulas (219) acquire meaning in the RHS given by (211) and (212). If we use the duality formula

$$\langle \mathscr{E}\phi_\pm(E)|F_\pm\rangle = \langle \phi_\pm(E)|\mathscr{E}F_\pm\rangle, \quad \forall \phi_\pm(E) \in S \cap \mathscr{H}^2_\pm, \quad \forall F_\pm \in (S \cap \mathscr{H}^2_\pm)^\times \tag{220}$$

we extend the action of the operator $\mathscr{E}$ to the duals $(S \cap \mathscr{H}^2_\pm)^\times$ and, then, (219) becomes meaningful.

Now, we are in the position to give a definition for the mean value of the energy for the Gamow vectors using this representation. Let us normalize ψ^{G} and ψ^{D} first and redefine

$$\psi^{\mathrm{G}} = \alpha \frac{1}{E - z_{\mathrm{R}}^*}, \quad \psi^{\mathrm{D}} = \alpha \frac{1}{E - z_{\mathrm{R}}} \ , \tag{221}$$

where α is a complex constant to be fixed by the normalization condition

$$\|\psi^{\mathrm{G}}\|^2 = \|\psi^{\mathrm{D}}\|^2 = \alpha^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{(E - E_{\mathrm{R}})^2 + (\Gamma/2)^2} = \alpha^2 \pi = 1 \ . \tag{222}$$

Therefore, $\alpha = 1/\sqrt{\pi}$.

Once we have chosen this normalization criterium, we can propose expressions for the mean value of the energy on the Gamow vectors:

$$\langle \psi^{\mathrm{G}} | \mathscr{E} | \psi^{\mathrm{G}} \rangle \quad \text{and} \quad \langle \psi^{\mathrm{D}} | \mathscr{E} | \psi^{\mathrm{D}} \rangle \ . \tag{223}$$

Let us evaluate these values. Since $\mathscr{E}\,\psi^{\mathrm{D}}(E) = E\,\psi^{\mathrm{D}}(E)$, we have that

$$\langle \psi^{\mathrm{D}} | \mathscr{E} | \psi^{\mathrm{D}} \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{E - z_{\mathrm{R}}^*} \frac{E}{E - z_{\mathrm{R}}} \mathrm{d}E = \frac{2}{\pi\Gamma} \int_{-\infty}^{\infty} \frac{E\,\mathrm{d}E}{\left(\frac{E - E_{\mathrm{R}}}{\Gamma/2}\right)^2 + 1} \ . \tag{224}$$

The change of variables

$$x = \frac{E - E_{\mathrm{R}}}{\Gamma/2} \ , \tag{225}$$

transforms the last integral in (224) into

$$\frac{E_{\mathrm{R}}}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + 1} + \frac{\Gamma}{2\pi} \int_{-\infty}^{\infty} \frac{x\,\mathrm{d}x}{x^2 + 1} \ . \tag{226}$$

The first integral in (226) has the value π. The second admits a Cauchy principal value equal to zero. Thus, we find

$$\langle \psi^{\mathrm{D}} | \mathscr{E} | \psi^{\mathrm{D}} \rangle = E_{\mathrm{R}} \tag{227}$$

and

$$\langle \psi^{\mathrm{G}} | \mathscr{E} | \psi^{\mathrm{G}} \rangle = E_{\mathrm{R}} \ . \tag{228}$$

Remark. (a) Observe that the integral in (224) does not converge although its Cauchy mean value does exists. Therefore, identities (227) and (228) are indeed Cauchy mean values.

(b) We see that this definition of the energy average of Gamow vectors gives the same real value for both Gamow vectors and coincides with the resonant energy. In addition, due to the adopted normalization, we have that $\langle \psi^{\mathrm{D}} | \psi^{\mathrm{D}} \rangle = \langle \psi^{\mathrm{G}} | \psi^{\mathrm{G}} \rangle = 1$. Furthermore,

$$\langle \psi^{\mathrm{G}} | \psi^{\mathrm{D}} \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{(E - z_{\mathrm{R}})^2} = 0 \ . \tag{229}$$

Analogously, $\langle \psi^{\mathrm{D}} | \psi^{\mathrm{G}} \rangle = 0$.

4.4. Real average values: Berggren interpretation

Berggren's approach to the mean value of the Hamiltonian on a Gamow state [18] can be formulated in a manner which is very similar to Bohm's. Following [33,34], we shall not use Hardy functions to construct the Gelfand triplets. Instead, we consider here another triplet $\tilde{\xi} \subset \mathscr{H} \subset \tilde{\xi}^{\times}$ for which the space $\tilde{\xi}^{\times}$ consists of tempered ultra-distributions [33,34,107]. See Section 3.9.

One of the advantages of this presentation is that the Gamow vectors are normalized in the sense that the product $\langle \tilde{f}_0 | f_0 \rangle$ exists and is given by [33,34]: [28]

$$\langle \tilde{f}_0 | f_0 \rangle = \frac{1}{4\pi^2 \Gamma} \left[\arctan\left(\frac{E_1 - E_R}{\Gamma} \right) - \arctan\left(\frac{E_0 - E_R}{\Gamma} \right) \right] . \tag{230}$$

This normalization allows also to define a mean value for the Hamiltonian in the form [33,34]:

$$\langle \tilde{f}_0 | H | f_0 \rangle = E_R + \frac{\Gamma}{2} \frac{\ln \left[\frac{(E_1 - E_R)^2 + \Gamma^2}{(E_0 - E_R)^2 + \Gamma^2} \right]}{\left[\arctan\left(\frac{E_1 - E_R}{\Gamma} \right) - \arctan\left(\frac{E_0 - E_R}{\Gamma} \right) \right]} . \tag{231}$$

Note that this mean value is *real*. Also, if we take *simultaneously* the limits $E_0 \to -\infty$ and $E_1 \to +\infty$, we have

$$\langle \tilde{f}_0 | H | f_0 \rangle = E_R . \tag{232}$$

We can also define the probability distribution associated the Gamow vector as

$$\begin{aligned} P(E) &:= |\langle E | f_0 \rangle|^2 \\ &= \frac{\Gamma}{(E - E_R)^2 + \Gamma^2} \cdot \frac{1}{\left[\arctan\left(\frac{E_1 - E_R}{\Gamma} \right) - \arctan\left(\frac{E_0 - E_R}{\Gamma} \right) \right]} . \end{aligned} \tag{233}$$

In the limit $E_1 \to +\infty, E_0 \to -\infty$ the above equation yields

$$P(E) = \frac{\Gamma/\pi}{(E - E_R)^2 + \Gamma^2} \tag{234}$$

which is the Breit–Wigner form proposed by [20,18].

The coincidence between (232) for the mean value of the Gamow states and Bohm's one, presented in the previous sub-section, comes from a re-interpretation of Berggren's definition given in [18]. In fact, for a spherically symmetric potential and for an arbitrary value of the angular momentum l, we can write the normalized decaying Gamow vector as

$$|f_0\rangle = \mathrm{i} \sqrt{\frac{2\Gamma}{\pi}} \int_0^{\infty} \sqrt{\frac{k}{m}} \frac{|k, \hat{k}, l\rangle}{E(\mathbf{k}) - z_R} \, \mathrm{d}k , \tag{235}$$

where, $k = |\mathbf{k}|$, $E(\mathbf{k}) = k^2/2m$ and $\hat{k}$ is the unit vector in the direction of $\mathbf{k}$.

A similar definition was advanced by Romo [103]

For the growing Gamow vector, we have

$$|\tilde{f}_0\rangle = -\mathrm{i} \sqrt{\frac{2\Gamma}{\pi}} \int_0^{\infty} \sqrt{\frac{k}{m}} \frac{|k, \hat{k}, l\rangle}{E(\mathbf{k}) - z_R^*} \, \mathrm{d}k . \tag{236}$$

[28] This normalization is not unique. In Ref. [13], Antoniou and Prigogine have proposed $\langle \tilde{f}_0 | f_0 \rangle = 1$.

Now, let A be an arbitrary observable. We can define the mean value of A on $|f_0\rangle$ as

$$\langle f_0|A|f_0\rangle = \frac{2\Gamma}{\pi}\sum_{l,l'}\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}k'\,\frac{\sqrt{kk'}}{m}\,\frac{\langle k',\hat{k}',l'|A|k,\hat{k},l\rangle}{(E(\mathbf{k}')-z_\mathrm{R})(E(\mathbf{k})-z_\mathrm{R}^*)}\ . \tag{237}$$

If we replace A by H we obtain, straightforwardly, the value E_R for this average. Instead, Berggren defines [18] this mean value as $\mathrm{Real}\langle\tilde{f}_0|A|f_0\rangle$. As a matter of fact, one can easily show that

$$\langle f_0|A|f_0\rangle = \mathrm{Real}\{\langle\tilde{f}_0|A|f_0\rangle\} + o(\Gamma^2)\ , \tag{238}$$

which means that Berggren's approximation coincides with Bohm's to the first order in Γ.

5. Relativistic Gamow vectors

To complete our presentation on Gamow vectors, and in order to introduce the reader in the field of relativistic resonances, we shall briefly comment on two of the existing approaches. One of them deals with resonances that appear in the interactions between relativistic fields. This approach has been developed by the school of Brussels, which has produced several specific models [6,7,67,9]. These models have in common that they can be solved by using a transformation that shows their equivalence with an exactly soluble Friedrichs model. We want to summarize here one of these models, in which a local Klein–Gordon field, with fixed mass, interacts with a bilocal Klein–Gordon field with an unbounded mass spectrum. The interaction is quadratic, so that it is exactly soluble. When the mass of the local field and the threshold of the mass spectrum for the bilocal field fulfill a simple relationship, the interaction is unstable and a resonance appears. Then, the corresponding Gamow vectors can be determined exactly as functionals in a RHS [7].

The other approach was developed by Bohm and collaborators [24,29–31] and it uses the properties of the relativistic S-matrix. In order to avoid the complex momenta that appear in the Brussels approach, due to Poincaré transformations over four-momenta with complex masses, Bohm et al. have introduced velocities. Although Bohms approach may look quite different to the Brusselss one, it is more general in the sense that it does not rely on specific models, i.e., it does not use of field theory. Nevertheless, the conclusions reached by both methods are essentially the same.

In the next subsection we shall focus on the discussion of the Brusselss method, since it is more closely related to the formalism which we have presented in this report. Since a detail comparison between Bohms and Brusselss methods may be out of the scope of the present work we leave it for a future effort.

5.1. An exactly solvable model for unstable relativistic quantum fields

This model involves a quadratic interaction between a local scalar field $\varphi(x)$ and a bilocal scalar field ψ. The interaction makes the local scalar field $\varphi(x)$ unstable.

The local scalar field $\varphi(x)$ with mass M may be written in terms of creation and annihilation operators as follows:

$$\varphi(\mathbf{x},t) = \int \mathrm{d}\tilde{\mathbf{k}}[a^\dagger(\mathbf{k})\,\mathrm{e}^{\mathrm{i}k\cdot x} + a(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}k\cdot x}]\ . \tag{239}$$

By boldface letters, we denote three dimensional vectors. Four-dimensional vectors in Minkowski space are denoted by roman style letters. The products of two vectors in Minkowski space are characterized by a dot. In Minkowski space we use the metric $(+---)$. For example, $k\cdot x=k_0x_0-\mathbf{kx}$. The Lorentz invariant measure in (239) is

$$\mathrm{d}\tilde{\mathbf{k}} = \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2\omega(\mathbf{k})}, \quad \omega(\mathbf{k}) = (\mathbf{k}^2 + M^2)^{1/2} \ . \tag{240}$$

The creation and annihilation operators in (239) satisfy the usual commutation relations:

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega(\mathbf{k})\, \delta(\mathbf{k}-\mathbf{k}') \ . \tag{241}$$

The Hamiltonian of the field $\varphi(x)$ is given by

$$H_M = \int \mathrm{d}\tilde{\mathbf{k}}\omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) \ . \tag{242}$$

We shall consider, in addition, a simple bilocal scalar field, $\psi(x_\mu, q)$, with continuous mass spectrum. The notion of bilocal field was introduced long ago by Yukawa [115] and Markov [83] in their discussion of the extended elementary particles. It is used here in a different context. The field depends also on an additional real variable q representing an internal degree of freedom. We shall start by considering the classical field and then quantize it. For simplicity we shall assume that the classical field $\psi(x_\mu, q)$ is an even function of q. This means that

$$\psi(x_\mu, -q) = \psi(x_\mu, q) \ . \tag{243}$$

For the mass operator $\mathbf{M}$, we choose the simplest possible form:

$$\mathbf{M}^2 = 4m^2 - \frac{\partial^2}{\partial q^2} \ . \tag{244}$$

The spectrum of the mass operator is $[2m, \infty)$. The field $\psi(x_\mu, q)$ satisfies the following generalized Klein–Gordon equation:

$$(\Box - \mathbf{M}^2)\psi(x_\mu, q) = 0 \ , \tag{245}$$

where $\Box$ is the usual d'Alambert operator. The solution of Eq. (245) can be written in the following form:

$$\psi(x_\mu, q) = \int \mathrm{d}^4 k_\mu \int_{-\infty}^{\infty} \mathrm{d}\kappa \, \frac{\cos q\kappa}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}k_\mu x^\mu} \delta(k^2 - 4m^2 - \kappa^2) B(k_\mu, \kappa) \ , \tag{246}$$

where we took into account (243). The amplitude $B(k_\mu, \kappa)$ is also an even function of κ. Integrating the r.h.s. of (246) over k_0 gives

$$\psi(x_\mu, q) = \int_{-\infty}^{\infty} \mathrm{d}\kappa \frac{\mathrm{d}^3\mathbf{k}\cos\kappa q}{(2\pi)^4 2E(\mathbf{k}, \kappa)} (B^*(\mathbf{k}, \kappa)\mathrm{e}^{\mathrm{i}k\cdot x} + B(\mathbf{k}, \kappa)\mathrm{e}^{-\mathrm{i}k\cdot x}) \ , \tag{247}$$

where $k_\mu = (E, \mathbf{k})$ and

$$E(\mathbf{k}, \kappa) = [4m^2 + \kappa^2 + \mathbf{k}^2]^{1/2} \ . \tag{248}$$

We can change the variables in (247) so that E is the new independent variable instead of κ:

$$\kappa = (E^2 - \mathbf{k}^2 - 4m^2)^{1/2}, \quad \frac{\mathrm{d}\kappa}{E} = \frac{\mathrm{d}E}{\kappa} \ , \tag{249}$$

then,

$$\psi(x_\mu, q) = \int_0^\infty \frac{\mathrm{d}E\, \mathrm{d}^3\mathbf{k} \cos\kappa(k_\mu)q}{(3\pi)^4\, \kappa(k_\mu)} (B^*(\mathbf{k}, E)\mathrm{e}^{\mathrm{i}k\cdot x} + B(\mathbf{k}, E)\mathrm{e}^{-\mathrm{i}k\cdot x}) \ . \tag{250}$$

Once we have the classical field written as in (250), we can make use of the standard quantization rules [19,106] to obtain the quantum field

$$\psi(x_\mu, q) = \int_0^\infty \frac{\mathrm{d}E\, \mathrm{d}^3\mathbf{k} \cos\kappa(k_\mu)q}{(2\pi)^4\, \kappa(k_\mu)} [B^\dagger(\mathbf{k}, E)\mathrm{e}^{\mathrm{i}k\cdot x} + B(\mathbf{k}, E)\mathrm{e}^{-\mathrm{i}k\cdot x}] \ . \tag{251}$$

where the creation $B^\dagger(\mathbf{k}, E)$ and annihilation operators $B(\mathbf{k}, E)$ satisfy the following commutation relations:

$$[B(\mathbf{k}, E), B^\dagger(\mathbf{k}', E')] = (2\pi)^4 \kappa(k_\mu)\delta^4(k_\mu - k'_\mu) \ . \tag{252}$$

The Hamiltonian for the bilocal field $\psi(x_\mu, q)$ is given by

$$H_m = \int \frac{\mathrm{d}^3\mathbf{k}\, \mathrm{d}E}{(2\pi)^4 \kappa(\mathbf{k}, E)} EB^\dagger(\mathbf{k}, E)B(\mathbf{k}, E) \ . \tag{253}$$

We now introduce the quadratic interaction Hamiltonian as

$$H_{\mathrm{int}} = -\lambda \int \mathrm{d}^3\mathbf{x} \int_{-\infty}^{\infty} \mathrm{d}q\, \psi(x, q) f(q) \varphi(x) \ , \tag{254}$$

where we assume that the even function $f(q)$ is a Lorentz scalar and has the Fourier transform:

$$f(q) = \int \mathrm{d}y\, \alpha(y) \cos yq \ . \tag{255}$$

The function $f(q)$ plays the same role as the form factor in the Friedrichs model. To avoid divergencies, we choose it so that it has a good asymptotic behavior. With this choice of the interaction, the total Hamiltonian becomes

$$\begin{aligned} P_0 = & \int \frac{\mathrm{d}^3\mathbf{k}\, \mathrm{d}E}{(2\pi)^4 \kappa(\mathbf{k}, E)} EB^\dagger(\mathbf{k}, E)B(\mathbf{k}, E) \\ & + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})} \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) + \int \frac{\mathrm{d}^3\mathbf{k}\, \mathrm{d}E}{(2\pi)^3 2\omega} \frac{\lambda\alpha(\kappa(\mathbf{k}, E))}{\kappa(\mathbf{k}, E)} \\ & \times (a(\mathbf{k}) + a^\dagger(-\mathbf{k}))(B^\dagger(\mathbf{k}, E) + B(-\mathbf{k}, E)) \ . \end{aligned} \tag{256}$$

Note that the function $\alpha(\cdot)$ in the third integral in (256) coincides with the function $\alpha(\cdot)$ that appears in (255). The three momentum for the interaction field is given by

$$\mathbf{P} = \int \frac{\mathrm{d}^3\mathbf{k}\, \mathrm{d}E}{(2\pi)^4 \kappa(\mathbf{k}, E)} \mathbf{k} B^\dagger(\mathbf{k}, E)B(\mathbf{k}, E) + \int \frac{\mathrm{d}^3\mathbf{k}\, \mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})} a^\dagger(\mathbf{k}) a(\mathbf{k}) \ . \tag{257}$$

The analogy between the Hamiltonian (256) and the interacting Hamiltonian in the usual Friedrichs model is obvious. As a matter of fact, we are considering here an *infinite collection* of Friedrichs models, each corresponding to a value of the momentum $\mathbf{k}$.

Our next step is to *diagonalize* the four momentum (256) and (257). This means that we are looking for creation, $b^\dagger(E,\mathbf{k})$, and annihilation, $b(E,\mathbf{k})$, operators such that the four momentum components P_μ can be written as

$$P_\mu = \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})}\, k_\mu b^\dagger(E,\mathbf{k})b(E,\mathbf{k}) \ , \tag{258}$$

where κ is given in (249). To achieve it, we write the eigenvalue equation [7]:

$$[P_\mu, b^\dagger(E,\mathbf{k})] = k_\mu b^\dagger(E,\mathbf{k}) \ , \tag{259}$$

where $k_\mu = (E,\mathbf{k})$ and the P_μ are given by (256) and (257). To solve (259), i.e., to obtain the creation operators $b^\dagger(E,\mathbf{k})$, we make the following ansatz:

$$\begin{aligned} b^\dagger(E,\mathbf{k}) = \int \mathrm{d}E'(&T(E,E',\mathbf{k})B^\dagger(E',\mathbf{k}) + R(E,E',\mathbf{k})B(E',-\mathbf{k})) \\ &+ t(E,\mathbf{k})a^\dagger(\mathbf{k}) + r(E,\mathbf{k})a(-\mathbf{k}) \ , \end{aligned} \tag{260}$$

which obviously means that we are assuming that $b^\dagger(E,\mathbf{k})$ is a linear combination of $B^\dagger(E,\mathbf{k})$, $B(E,\mathbf{k})$; $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$. In order to obtain $b^\dagger(E,\mathbf{k})$, we insert (260) into (259) to obtain four equations in the undetermined amplitudes $T(E,E',\mathbf{k})$, $R(E,E',\mathbf{k})$, $t(E,\mathbf{k})$ and $r(E,\mathbf{k})$ [7].

The formal solution, for positive energies, of the eigenvalue equation (259) is [7]

$$\begin{aligned} b^\dagger(E,\mathbf{k}) = C\Bigg\{ &B^\dagger(E,\mathbf{k}) + 2\pi\lambda\alpha(\kappa(E,\mathbf{k}))G(E,\mathbf{k}) \\ &\times \Bigg[\int \mathrm{d}E' \frac{\lambda\alpha(\kappa(E',\mathbf{k}))}{\kappa(E',\mathbf{k})}(E',\mathbf{k}) \times \left(\frac{B^\dagger(E,\mathbf{k})}{E'-E} - \frac{B(E,-\mathbf{k})}{E'+E}\right) \\ &- \frac{(E+\omega(\mathbf{k}))a^\dagger(\mathbf{k}) + (E-\omega(\mathbf{k}))a(-\mathbf{k})}{2\omega(\mathbf{k})}\Bigg]\Bigg\} \ . \end{aligned} \tag{261}$$

Let us analyze this result. The functions $\omega(\mathbf{k})$, $\alpha(\cdot)$ and $\kappa(E,\mathbf{k})$ have been defined already in (240), (255) and (249), respectively. The most interesting object in (261) is the Green function $G(E,\mathbf{k})$, where

$$G(E,\mathbf{k}) = \frac{1}{\omega^2 - E^2 - \Pi(E,\mathbf{k})} \tag{262}$$

with

$$\Pi(E,\mathbf{k}) = \int_{E_0}^{\infty} \mathrm{d}E' 2E' \frac{\rho(E',\mathbf{k})}{E'^2 - E^2} \ , \tag{263}$$

and $E_0 = (4m^2 + \mathbf{k}^2)^{1/2}$. Therefore, $G(E,\mathbf{k})$ depends on

$$\rho(E,\mathbf{k}) = 2\pi \frac{\lambda^2\alpha^2(\kappa(E,\mathbf{k}))}{\kappa(E,\mathbf{k})} \ , \tag{264}$$

through $\Pi(E,\mathbf{k})$, since $\rho(E,\mathbf{k})$ is given by Eq. (264). Note that $\rho(E,\mathbf{k})$ depends on $\alpha(\cdot)$ and hence on the form factor $f(q)$.

The properties of $G(E,\mathbf{k})$ and $\Pi(E,\mathbf{k})$ are discussed in [7]. Formulas (233) and (234) are also valid for complex values of E and the Green function $G(E,\mathbf{k})$ is analytic in E^2 with a branch cut in $[E_0^2,\infty)$. The Green function admits analytic continuations from above to below $G(E+\mathrm{i}0,\mathbf{k})$, i.e., from the upper to the lower half plane, and from below to above $G(E-\mathrm{i}0,\mathbf{k})$ through the cut. If $M\geqslant 2m$, $G(E+\mathrm{i}0,\mathbf{k})$ has a pole at $E^2=\mathbf{k}^2+\mu_c^2$ and $G(E-\mathrm{i}0,\mathbf{k})$ has a pole at the conjugate point $E^2=\mathbf{k}^2+\mu_c^{*2}$, where:

$$\mu_c^2=\mu^2-\mathrm{i}\mu\Gamma\ , \tag{265}$$

and μ and Γ are real positive numbers which depend on the form factor $f(q)$ [7].

The equation for the complex pole μ_c^2 is:

$$\omega^2(\mathbf{k})-E^2-\int \mathrm{d}E'^2\,\frac{\rho(E',\mathbf{k})}{E'^2-E^2}=0\ . \tag{266}$$

Compare this equation with (140). Eq. (266) can be written as

$$M^2-(E^2-\mathbf{k}^2)-\mathrm{PV}\int \mathrm{d}E'^2\,\frac{\rho(E',\mathbf{k})}{E'^2-E^2}-\mathrm{i}\pi\rho(E,\mathbf{k})=0\ , \tag{267}$$

where PV stands for the Cauchy principal value. For small values of the coupling constant λ, we can omit the integral term and the result is

$$\mu_c^2=M^2-\mathrm{i}\pi\rho(E,\mathbf{k})|_{E^2-\mathbf{k}^2=M^2}=M^2-2\mathrm{i}\pi^2\lambda^2\,\frac{[\alpha(\sqrt{M^2-4m^2})]^2}{\sqrt{M^2-4m^2}}\ . \tag{268}$$

From (268), we note that the pole appears if $M\geqslant 2m$ only.

Let us go back to the formal solution (261). Along this solution of (259) for positive energies, there exists another formal solution for negative energies which is given by [7]

$$b(E,\mathbf{k})=C\left\{B(E,\mathbf{k})+2\pi\lambda\alpha(\kappa(E,\mathbf{k}))G(E,\mathbf{k})\left[\int \mathrm{d}E'\,\frac{\lambda\alpha(\kappa(E,\mathbf{k}))}{\kappa(E,\mathbf{k})}\right.\right.$$
$$\left.\left.\times\left(\frac{B(E,\mathbf{k})}{E'-E}-\frac{B^+(E,-\mathbf{k})}{E'+E}\right)-\frac{(E+\omega(\mathbf{k}))a(\mathbf{k})+(E-\omega(\mathbf{k}))a^+(-\mathbf{k})}{2\omega(\mathbf{k})}\right]\right\}\ . \tag{269}$$

This is the annihilation operator.

The signs of the boundary conditions $G(E\pm\mathrm{i}0,\mathbf{k})$ that provide both analytic continuations of the Green function, coincide with the signs in the denominator of the Lippmann–Schwinger equations [90]. These denominators have the sign plus for the incoming and the sign minus for the outgoing equation respectively. Therefore, the solution of the eigenvalue equation (230) with *incoming* boundary conditions is obtained by replacing $G(E,\mathbf{k})$ by $G(E+\mathrm{i}0,\mathbf{k})$ in (261) and (269). The *incoming operators* are

$$b_{\mathrm{in}}^\dagger(E,\mathbf{k})=B^\dagger(E,\mathbf{k})+2\pi\lambda\alpha(\kappa(E,\mathbf{k}))G(E+\mathrm{i}0,\mathbf{k})\left[\int \mathrm{d}E'\,\frac{\lambda\alpha(\kappa(E',\mathbf{k}))}{\kappa(E',\mathbf{k})}\right.$$
$$\left.\times\left(\frac{B^\dagger(E,\mathbf{k})}{E'-E-\mathrm{i}0}-\frac{B(E',-\mathbf{k})}{E'+E}\right)-\frac{(E+\omega(\mathbf{k}))a^\dagger(\mathbf{k})+(E-\omega(\mathbf{k}))a(-\mathbf{k})}{2\omega(\mathbf{k})}\right] \tag{270}$$

and

$$b_{\rm in}(E,\mathbf{k}) = B(E,\mathbf{k}) + 2\pi\lambda\alpha(\kappa(E,\mathbf{k}))G(E-{\rm i}0,\mathbf{k})\left[\int {\rm d}E'\,\frac{\lambda\alpha(\kappa(E',\mathbf{k}))}{\kappa(E',\mathbf{k})}\right.$$
$$\left.\times\left(\frac{B(E',\mathbf{k})}{E'-E+{\rm i}0}-\frac{B^\dagger(E',-\mathbf{k})}{E'+E}\right)-\frac{(E+\omega(\mathbf{k}))a(\mathbf{k})+(E-\omega(\mathbf{k}))a^\dagger(-\mathbf{k})}{2\omega(\mathbf{k})}\right]\ . \quad (271)$$

Here,

$$E\geqslant E_0(\mathbf{k}) = (4m^2+\mathbf{k}^2)^{1/2}$$

and we have chosen the constant $C=1$ in (268). The *outgoing operators*, $b^\dagger_{\rm out}(E,\mathbf{k})$ and $b_{\rm out}(E,\mathbf{k})$, are obtained with the change $+{\rm i}0\mapsto -{\rm i}0$ in (270) and (271), respectively. These operators have the following commutation relations [7]:

$$[b_{\rm in}(E,\mathbf{k}),b^\dagger_{\rm in}(E',\mathbf{k}')] = [b_{\rm out}(E,\mathbf{k}),b^\dagger_{\rm out}(E',\mathbf{k})]\ ,$$
$$= (2\pi)^4\kappa(E,\mathbf{k})\delta(E-E')\delta^3(\mathbf{k}-\mathbf{k}')\ ,$$
$$[b_{\rm out}(E,\mathbf{k}),b^\dagger_{\rm in}(E',\mathbf{k}')] = (2\pi)^4\kappa(E,\mathbf{k})\delta(E-E')\delta^3(\mathbf{k}-\mathbf{k}')\,\frac{G(E+{\rm i}0,\mathbf{k})}{G(E-{\rm i}0,\mathbf{k})}\ ,$$
$$[b_{\rm in}(E,\mathbf{k}),b^\dagger_{\rm out}(E',\mathbf{k}')] = (2\pi)^4\kappa(E,\mathbf{k})\delta(E-E')\delta^3(\mathbf{k}-\mathbf{k}')\,\frac{G(E-{\rm i}0,\mathbf{k})}{G(E+{\rm i}0,\mathbf{k})}\ . \quad (272)$$

All other commutators vanish. These operators are the solutions to the diagonalization problem in the sense of Eq. (258) holds, i.e.,

$$P_\mu = \int\frac{{\rm d}^3\mathbf{k}\,{\rm d}E}{(2\pi)^4\kappa(E,\mathbf{k})}\,k_\mu b^\dagger_{\substack{\rm in\\ \rm out}}(E,\mathbf{k})b_{\substack{\rm in\\ \rm out}}(E,\mathbf{k})\ . \quad (273)$$

Now, we are ready to obtain the Gamow vectors for the unstable field.

5.2. Resonances and Gamow vectors

Let us start with the following remark [7]: Let us consider the vacuum state before the interaction, is switched on. It is characterized by the following equations:

$$B(E,\mathbf{k})|0\rangle = 0 \quad \text{and} \quad a(\mathbf{k})|0\rangle = 0\ . \quad (274)$$

This is not the vacuum for the interacting field, since [7]

$$b(E,\mathbf{k})|0\rangle \neq 0\ , \quad (275)$$

i.e. there arise a new vacuum state which will be a superposition of states with an arbitrary number of particles of B and a-types. The new vacuum Ω can be obtained from the old one by means of a transformation of the type

$$|\Omega\rangle = {\rm e}^V|0\rangle\ , \quad (276)$$

where V is a quadratic functional of creation operators $B^\dagger(E,\mathbf{k})$ and $a^\dagger(\mathbf{k})$. It results that

$$b_{\rm in}(E,\mathbf{k})|\Omega\rangle = 0\ . \quad (277)$$

Due to the form (269) of $b^\dagger_{\rm in}(E,\mathbf{k})$, the state

$$\Phi_{\rm in}(E,\mathbf{k}) = b^\dagger_{\rm in}(E,\mathbf{k})|\Omega \tag{278}$$

has a pole as a function of E when we make the analytic continuation from above to below. This pole has been found to be at the point [7]

$$z_{\rm R} = (\mathbf{k} + \mu^2 - {\rm i}\mu\Gamma)^{1/2} \ , \tag{279}$$

and the width Γ and therefore the lifetime depend on $\mathbf{k}$. The state $\Phi_{\rm in}(E,\mathbf{k})$ can be written in a neighborhood of $z_{\rm R}$ as

$$\Phi_{\rm in}(E,\mathbf{k}) = \frac{1}{E - z_{\rm R}}\,\varphi^{\rm G}_{\rm in}(\mathbf{k}) + \text{regular part} \ . \tag{280}$$

The residue $\varphi^{\rm G}_{\rm in}(\mathbf{k})$ has the following properties [7]:

(i) It is an eigenvector of the total Hamiltonian P_0 with eigenvalue $z_{\rm R}$,

$$P_0\varphi^{\rm G}_{\rm in}(\mathbf{k}) = z_{\rm R}\varphi^{\rm G}_{\rm in}(\mathbf{k}) \ . \tag{281}$$

Therefore $\varphi^{\rm G}_{\rm in}(\mathbf{k})$ is the Gamow vector associated to the resonance with resonance pole at $z_{\rm R}$.

(ii) As a consequence of (i), $\varphi^{\rm G}_{\rm in}(\mathbf{k})$ decays exponentially:

$$\mathrm{e}^{-{\rm i}tP_0}\varphi^{\rm G}_{\rm in}(\mathbf{k}) = \mathrm{e}^{-{\rm i}tz_{\rm R}}\varphi^{\rm G}_{\rm in}(\mathbf{k}) \ . \tag{282}$$

Properties (i) and (ii) show that $\varphi^{\rm G}_{\rm in}(\mathbf{k})$ cannot be a normalizable vector in a Hilbert (Fock) space. As happens in the non-relativistic case, we need to rig the Fock space in order to find the true nature of $\varphi^{\rm G}_{\rm in}(\mathbf{k})$. There is a standard procedure, [19,54,94] and this is done in three steps:

(i) We choose the one particle "test vector space", that we call $\boldsymbol{\Phi}$. The space $\boldsymbol{\Phi}$ has sufficient properties so that $\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^\times$ is a RHS, where $\mathscr{H}$ is the Hilbert space of one particle states.

(ii) Starting with $\boldsymbol{\Phi}$, we construct the test vector space for arbitrary number of particles. This space is the Fock algebra given by

$$\mathfrak{F} = \mathbb{C} \oplus \boldsymbol{\Phi} \oplus (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi}) \oplus (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi} \otimes \boldsymbol{\Phi}) \oplus \dots \ , \tag{283}$$

where $\mathbb{C}$ is the set of complex numbers and $\oplus$ and $\otimes$ means direct sum and tensor product, respectively. This Fock algebra admits a topology which is obtained from the topology on $\boldsymbol{\Phi}$ [54,7] and that is stronger (has more open sets) than the topology of the Hilbert space on the Fock space

$$\mathscr{H}^{\otimes} = \mathbb{C} \oplus \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H}) \oplus (\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}) \dots \ .$$

(iii) Then, the dual $\mathfrak{F}^\times$ of $\mathfrak{F}$ is obtained. We have the RHS

$$\mathfrak{F} \subset \mathscr{H}^{\otimes} \subset \mathfrak{F}^\times \tag{284}$$

In our particular case, we construct $\mathfrak{F}$ as follows: first, we choose the following space of *test functions*

$$\boldsymbol{\Psi} = (\mathscr{H}^2_- \cap S) \otimes S(\mathbb{R}^3) \ , \tag{285}$$

where

(i) $\mathscr{H}^2_-$ is the space of Hardy functions on the lower half-plane.
(ii) S is the Schwartz space described in Section 2.7.

(iii) $S(\mathbb{R}^3)$ represents the space of all complex functions on the three dimensional real space $\mathbb{R}^3$ having the same properties as S in (ii). Usually, we call S and $S(\mathbb{R}^3)$ the one-dimensional and the three-dimensional Schwartz space, respectively.

(iv) The topology on $\boldsymbol{\Psi}$ is then obtained from the topologies on the Schwartz spaces. This is a technical matter [93], but this topology makes both $\boldsymbol{\Psi}$ and $\mathfrak{F}$ metric spaces, i.e., the topology can be obtained from a metric.

A typical function of $(\mathscr{H}^2_- \cap S)\otimes S(\mathbb{R}^3)$ is of the form $\varphi(E,\mathbf{k})$. For a fixed value of the momentum $\mathbf{k}$, this is a function of the energy and belongs to $\mathscr{H}^2_- \cap S$. For a fixed value of the energy E, $\varphi(E,\mathbf{k}) \in S(\mathbb{R}^3)$.

Thus, if $\varphi(E,\mathbf{k})$ is in the space $(\mathscr{H}^2_- \cap S) \otimes S(\mathbb{R}^3)$, then, $\varphi(E,\mathbf{k})$ represents an one particle state. This state is equally well represented by the bra vector

$$\langle\varphi| = \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})}\langle\Omega|b_{\mathrm{out}}(E,\mathbf{k})\varphi(E,\mathbf{k}) \ . \tag{286}$$

The mapping given by

$$\varphi(E,\mathbf{k}) \mapsto \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})}\langle\Omega|b_{\mathrm{out}}(E,\mathbf{k})\varphi(E,\mathbf{k}) \ ,$$

is one to one and, hence, the space $\boldsymbol{\Phi}$ is the space of vectors of the form (286). Once we have $\boldsymbol{\Phi}$, we have the triplet (284). The action of the Gamow vector $\varphi^{\mathrm{G}}_{\mathrm{in}}(\mathbf{k})$ on the arbitrary vector of $\boldsymbol{\Phi}$ given by (286) is

$$\langle\varphi|\varphi^{\mathrm{G}}_{\mathrm{in}}(\mathbf{k})\rangle = \varphi(E,\mathbf{k})\left.\frac{c}{\gamma_-(-E)}\right|_{E=z_{\mathrm{R}}} \ , \tag{287}$$

where [7] $\gamma_\pm(E)$ are, respectively, the solutions of the equations given by

$$\gamma_\pm(E,\mathbf{k})\gamma_\pm(-E,\mathbf{k}) = G_\pm(E,\mathbf{k}) \ . \tag{288}$$

Note that the Gamow vector is a functional that vanishes on the vacuum and on the space of two or more particles.

This model is exactly solvable because the interaction between fields is quadratic. Due to the presence of a mass spectrum in the second field, the interaction contains non-trivial features like those studied here. Relativistic Gamow vectors can be obtained approximately when the interaction between fields is not quadratic [6].

6. Conclusions

In this work we have presented the essentials of the mathematical and physical interpretation of resonances. We have shown that the mathematical difficulties arising from the probabilistic interpretation of resonances, in the context of ordinary quantum mechanics, are easily removed by considering the resonances as vectors belonging to rigged Hilbert spaces. The chain of mathematical steps needed to reach this result has been discussed by: (i) defining Møller wave operators, (ii) analyzing the pole structure of the S-matrix, (iii) introducing the continuation of the scattering states in a larger representation, the rigged Hilbert space, where the resonant wave functions can be un-ambiguously

defined. The correctness of the approach was illustrated by the solutions of a schematic albeit realistic model (the Friedrisch model) which displays bound state and resonant features. We have focussed on the physical interpretation of mean values of observables on Gamow states, because this is of particular interest concerning realistic applications of the method. This is the case of the Berggren realization and its application in realistic nuclear structure calculations. The discussion of that part of our report was inspired by the work of Liotta and collaborators, who has advocated the use of Berggren resonant basis in the calculation of nuclear excitations. We have compared the results of different interpretations related to mean values on resonant states. The meaning of the procedure adopted by the Stockholm group, to incorporate isolated Gamow resonances to account for continuum nuclear structure effects, starting from single particle states which are solutions of the nuclear central potential with purely outgoing boundary conditions, was discussed in connection with the identification of resonances. We reported on the correctness of this approach, too, and give some examples of the adequate boundary conditions. Finally, we have discussed the extension of the formalism to the relativistic domain.

We would like to conclude by pointing out the prospects of the use of Gamow states in the microscopic description of nuclei far from the stability line, where, like in the case of α decay, resonant states may play a decisive role in the explanation of new decay modes (like two-proton radioactivity) and correlations (like pairing correlations in the continuum).

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Appendix A

At the end of Section 2.6, we have introduced the Gelfand–Maurin spectral theorem. Let us go back to formula (32) and omit the arbitrary vector $\varphi \in \boldsymbol{\Phi}$ and take $f(A) \equiv I$, the identity operator. Then, (32) leads to

$$\phi = \int_{\sigma(A)} |\lambda\rangle\langle\lambda|\phi\rangle \, \mathrm{d}\lambda \ . \tag{A.1}$$

The function $\phi(\lambda) = \langle\lambda|\phi\rangle = \langle\phi|\lambda\rangle^*$ is the wave function for the pure state ϕ in the A-representation. If B is another observable, the Gelfand–Maurin theorem for B implies the existence of a new RHS

$\boldsymbol{\Psi} \subset \mathscr{H} \subset \boldsymbol{\Psi}^{\times}$ such that B is continuous on $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}^{\times}$ and for all $\psi \in \boldsymbol{\Psi}$, we have

$$f(B)\psi = \int_{\sigma(B)} f(b)|b\rangle\langle b|\psi\rangle \, \mathrm{d}b \ , \tag{A.2}$$

where $b \in \sigma(B)$ and $B|b\rangle = b|b\rangle$, $|b\rangle \in \boldsymbol{\Psi}^{\times}$ and

$$\psi = \int_{\sigma(B)} |b\rangle\langle b|\psi\rangle \, \mathrm{d}b \ , \tag{A.3}$$

where $\psi(b) = \langle b|\psi\rangle = \langle\psi|b\rangle^*$ is the wave function of ψ in the B-representation. If further, $\psi \in \boldsymbol{\Phi}$, we use (A.3) in (A.1) to conclude that

$$\psi = \int_{\sigma(B)} \int_{\sigma(A)} |b\rangle\langle b|\lambda\rangle\langle\lambda|\psi\rangle \, \mathrm{d}b \, \mathrm{d}\lambda \ , \tag{A.4}$$

If $A = B$, obviously $\langle b|\lambda\rangle = \delta(\lambda - b)$. In the general case, $A \neq B$ and the kernel $\langle b|\lambda\rangle$ obviously depend on A and B. Multiplying (A.4) to the left by the bra $\langle\lambda|$, we obtain

$$\langle\lambda|\psi\rangle = \int_{\sigma(B)} \langle\lambda|b\rangle\langle b|\psi\rangle \, \mathrm{d}b \ . \tag{A.5}$$

which gives a direct relation between the wave functions in the B representation and in the A representation. This formula is invertible. If we multiply (A.1) to the left by the bra $\langle b|$, we obtain that

$$\langle b|\phi\rangle = \int_{\sigma(A)} \langle b|\lambda\rangle\langle\lambda|\phi\rangle \, \mathrm{d}\lambda \ . \tag{A.6}$$

A typical example is $A = Q$ and $B = P$ the one-dimensional position and momentum operators, respectively. We know that these two operators have purely continuous spectrum covering the real axis. Then, $Q|x\rangle = x|x\rangle$, $\forall x \in \mathbb{R}$ and $P|p\rangle = p|p\rangle$, $\forall p \in \mathbb{R}$. The kets $|x\rangle$ and $|p\rangle$ are antilinear continuous functionals on the Schwartz space S [26]. We also know that the Fourier transform changes the wave function for a pure state from the position representation into the momentum representation. Then, we have

$$\langle x|\psi\rangle = \int_{-\infty}^{\infty} \langle x|p\rangle\langle p|\psi\rangle \, \mathrm{d}p \ , \tag{A.7}$$

which implies that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-\mathrm{i}px} \ . \tag{A.8}$$

The Gelfand–Maurin theorem can be extended to a set of commuting observables. Here, we shall focus our attention to the case in which H, L^2 and L_z are a set of commuting observables, where $\vec{L} \equiv (L_x, L_y, L_z)$ represents the orbital angular momentum. Then, there exists a RHS $\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^{\times}$

such that the eigenvalue equations

$$H|E,l,l_z\rangle = E|E,l,l_z\rangle \ ,$$

$$L^2|E,l,l_z\rangle = l(l+1)|E,l,l_z\rangle \ ,$$

$$L_z|E,l,l_z\rangle = l_z|E,l,l_z\rangle \tag{A.9}$$

have solution in the antidual of the three dimensional Schwartz space $S(\mathbb{R}^3)$. Another possibility for a set of commuting observables is the triplet of components of the three-dimensional position, $\{Q_x, Q_y, Q_z\}$, or momentum operator, $\{P_x, P_y, P_z\}$. In these two latter cases, the eigenvalue equations

$$Q_x|\mathbf{x}\rangle = x|\mathbf{x}\rangle, \quad Q_y|\mathbf{x}\rangle = y|\mathbf{x}\rangle, \quad Q_z|\mathbf{x}\rangle = z|\mathbf{x}\rangle \ , \tag{A.10}$$

$$P_x|\mathbf{p}\rangle = p_x|\mathbf{p}\rangle, \quad P_y|\mathbf{p}\rangle = p_y|\mathbf{p}\rangle, \quad P_z|\mathbf{p}\rangle = p_z|\mathbf{p}\rangle \tag{A.11}$$

with $\mathbf{x} = (x, y, z)$ and $\mathbf{p} = (p_x, p_y, p_z)$, also have solutions in the antidual of $S(\mathbb{R}^3)$. Note that if $\psi(\mathbf{x}) \in S(\mathbb{R}^3)$, then in the H, L^2, L_z representation, the wave function for the state $\psi(\mathbf{x})$ is given by

$$\langle E,l,l_z|\psi\rangle = \int_{\mathbb{R}^3} \langle E,l,l_z|\mathbf{x}\rangle\langle\mathbf{x}|\psi\rangle \, \mathrm{d}^3\mathbf{x} \tag{A.12}$$

with $\langle\mathbf{x}|\psi\rangle = \psi(\mathbf{x})$. Eq. (A.12) is invertible:

$$\langle\mathbf{x}|\psi\rangle = \sum_{l_z=-l}^{l} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} \langle\mathbf{x}|E,l,l_z\rangle\langle E,l,l_z|\psi\rangle \, \mathrm{d}^3\mathbf{x} \tag{A.13}$$

Appendix B

Along this paper, we have defined Gamow vectors as eigenvectors of the Hamiltonian H with complex eigenvalues. These eigenvalues correspond to the poles of the analytic continuation of the S-operator in the energy representation. Their eigenvectors, the Gamow vectors, belong to the duals of a given RHS. However, the set of complex eigenvalues of the Hamiltonian on these duals is in general larger than the set of poles of the S-operator. For instance, the eigenvalue equation $H\psi = z\psi$ has solutions with ψ in $\boldsymbol{\Phi}_+^\times$ in (58) for all all z with negative imaginary part and in $\boldsymbol{\Phi}_-^\times$, also in (58), for all z with positive imaginary part. Thus, it would be interesting to have a criterion to know which solutions of the eigenvalue equation $H\psi = z\psi$ correspond to Gamow vectors, if we do not know the analytic structure of $S(E)$.

In order to make this criterion more comprehensible, let us present it on a simple model following the lines of previous work [27,79–82]. We study a resonant scattering model with a spherically symmetric potential of the type:

$$V(r) = \begin{cases} 0, & 0 < r < a \ , \\ V_0, & a \leqslant r \leqslant b \ , \\ 0, & b < r < \infty \ . \end{cases} \tag{B.1}$$

Let $\psi(x)$ be a pure quantum state in $S(\mathbb{R}^3)$. Using the Gelfand–Maurin theorem for a system of commuting observables, we have that

$$\psi = \sum_{l_z=-l}^{l} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} |E, l, l_z\rangle\langle E, l, l_z|\psi\rangle \, \mathrm{d}E \ . \tag{B.2}$$

If we apply H to (B.2), we have that

$$H\psi = \sum_{l_z=-l}^{l} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} H|E, l, l_z\rangle\langle E, l, l_z|\psi\rangle \, \mathrm{d}E \ . \tag{B.3}$$

Then,

$$\langle \mathbf{x}|H\psi\rangle = \sum_{l_z=-l}^{l} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} \langle \mathbf{x}|H|E, l, l_z\rangle\langle E, l, l_z|\psi\rangle \, \mathrm{d}E \ . \tag{B.4}$$

As $\langle \mathbf{x}|H\psi\rangle$ is the wave function in coordinate representation of $H\psi$, the Schrödinger equation yields

$$\langle \mathbf{x}|H\psi\rangle = \left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})\right)\langle \mathbf{x}|\psi\rangle \ , \tag{B.5}$$

where Δ is the three dimensional Laplacian. If we use (B.13) in (B.5), we have that

$$\langle \mathbf{x}|H\psi\rangle = \sum_{l_z=-l}^{l} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} \left\{\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})\right)\langle \mathbf{x}|E, l, l_z\rangle\right\}\langle E, l, l_z|\psi\rangle \, \mathrm{d}E \ , \tag{B.6}$$

where the brackets in (B.6) shows that only the kernel $\langle \mathbf{x}|E, l, l_z\rangle$ depends on the variable $\mathbf{x}$. Comparing (B.6)–(B.4) and taking into account that $\langle E, l, l_z|\psi\rangle$ is arbitrary, we conclude that

$$\langle \mathbf{x}|H|E, l, l_z\rangle = \left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})\right)\langle \mathbf{x}|E, l, l_z\rangle \ . \tag{B.7}$$

Since $V(\mathbf{x}) = V(r)$ is spherically symmetric, we can use spherical coordinates $\mathbf{x} = (r, \theta, \phi)$. Then, the Schrödinger equation (B.7) in spherical coordinates reads

$$\begin{aligned}\langle r, \theta, \phi|H|E, l, l_z\rangle &= \left(-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\langle r, \theta, \phi|E, l, l_z\rangle \\ &= E\langle r, \theta, \phi|E, l, l_z\rangle \ .\end{aligned} \tag{B.8}$$

If we separate the radial and angular dependence, we obtain

$$\langle r, \theta, \phi|E, l, l_z\rangle = \langle r|E\rangle_l\langle \theta, \phi|l, l_z\rangle = \frac{1}{r}\chi_l(r; E)Y_{l,l_z}(\theta, \phi) \ , \tag{B.9}$$

where $Y_{l,l_z}(\theta, \phi)$ are the spherical harmonics. Connecting (B.8) and (B.9), we obtain for the radial part:

$$\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\chi_l(r; E) = E\chi_l(r; E) \ . \tag{B.10}$$

Therefore, $\chi_l(r;E)$ is the solution to the radial part of the Schrödinger equation with orbital angular momentum equal to l. If we choose $l=0$ and write $\chi(r;E)=\chi_0(r;E)$, we get

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\chi(r;E)+V(r)\chi(r;E)=E\chi(r;E)\ . \tag{B.11}$$

The solutions of (B.11) are the Dirac kets, with fixed E, in the energy representation. The general solution of (B.11) is given by

$$\chi(r;E)=\begin{cases}\alpha_1\mathrm{e}^{\mathrm{i}kr}+\beta_1\mathrm{e}^{-\mathrm{i}kr}, & 0<r<a\ ,\\ \alpha_2\mathrm{e}^{\mathrm{i}Qr}+\beta_2\mathrm{e}^{\mathrm{i}Qr}, & a<r<b\ ,\\ \mathscr{F}_1\mathrm{e}^{\mathrm{i}kr}+\mathscr{F}_2\mathrm{e}^{-\mathrm{i}kr}, & b<r<\infty\ ,\end{cases} \tag{B.12}$$

where $\alpha_{1,2},\beta_{1,2}$ and $\mathscr{F}_{1,2}$ are arbitrary constants that depend on E and which we shall fix by using boundary conditions and

$$k=\sqrt{\frac{2m}{\hbar^2}E},\quad Q=\sqrt{\frac{2m}{\hbar^2}(E-V_0)}\ . \tag{B.13}$$

Taking into account the form of the potential, the boundary conditions are [79,80]:

$$\chi(0;E)=0\ , \tag{B.14}$$

i.e., the wave function vanishes at the origin, and

$$\begin{aligned}&\chi(a-0;E)=\chi(a+0;E)\ ,\\ &\chi'(a-0;E)=\chi'(a+0;E)\ ,\\ &\chi(b-0;E)=\chi(b+0;E)\ ,\\ &\chi'(b-0;E)=\chi'(b+0;E)\ ,\end{aligned} \tag{B.15}$$

which implies the continuity of the function and its first derivative at the points a and b.

After (B.11), the Dirac ket $\chi(r;E)$ satisfies the eigenvalue equation

$$H\chi(r;E)=E\chi(r;E)$$

for positive E. In order to look for Gamow vectors, we have to solve an eigenvalue equation of this type with complex E. Then the eigenvalue equation to consider must have the form

$$H\chi(r;z_0)=z_0\chi(r;z_0)\ ,$$

where z_0 is a complex number. For $l=0$, we have to solve the equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\chi(r;z_0)+V(r)\chi(r;z_0)=z_0\chi(r;z_0)\ . \tag{B.16}$$

The solution $\chi(r;z_0)$ of (B.16) has the form (B.12) with

$$k = k_0 = \sqrt{\frac{2m}{\hbar^2} z_0}, \quad Q = Q_0 = \sqrt{\frac{2m}{\hbar^2}(z_0 - V_0)} \ . \tag{B.17}$$

We impose to the solutions of (B.16) the boundary conditions:

$$\chi(0;z_0) = 0 \ , \tag{B.18}$$

i.e., the wave function vanishes at the origin;

$$\begin{aligned} &\chi(a-0;z_0) = \chi(a+0;z_0) \ , \\ &\chi'(a-0;z_0) = \chi'(a+0;z_0) \ , \\ &\chi(b-0;z_0) = \chi(b+0;z_0) \ , \\ &\chi'(b-0;z_0) = \chi'(b+0;z_0) \ , \end{aligned} \tag{B.19}$$

which implies the continuity of the function $\chi(r;z_0)$ and its first derivative, with respect to r, at the points a and b. Note that these boundary conditions differ from the boundary conditions given in Eqs. (B.14) and (B.15) of this Appendix B. At infinity, we impose the so called *purely outgoing boundary condition* [79,80], which is

$$\chi(r;z_0) \sim \mathrm{e}^{\mathrm{i}kr} \quad \text{for } k \mapsto \infty \ . \tag{B.20}$$

See also [71]. The meaning of this boundary condition is the following: take the latter formula of (B.19) and note that $\mathscr{F}_1$ and $\mathscr{F}_2$ must depend on k and, therefore, on z. Then, the S-matrix is written in the form [89,64]

$$S(k) = -\frac{\mathscr{F}_1(k)}{\mathscr{F}_2(k)} \ . \tag{B.21}$$

Resonances are placed at the poles of $S(k)$ for $\mathrm{Im}\, k < 0$, $\mathrm{Real}\, k \neq 0$ and this corresponds to the zeroes of $\mathscr{F}_2(k)$ with identical properties. Thus, if a resonance is placed at $z_0 = (\hbar^2 k_0^2)/(2m)$, then, $\mathscr{F}_2(k_0) = 0$ and reciprocally. This justifies characterization (B.20) for resonance states.

Thus, a solution of the eigenvalue equation $H\chi(r;z) = z\chi(r;z)$ represents a Gamow vector for the square well potential (B.1) and $l = 0$ if and only if it fulfills the purely outgoing boundary condition (B.20).

If $\chi(r;z_0)$ is a Gamow vector, it still may happens that either $\mathrm{Im}\, z_0 < 0$ or $\mathrm{Im}\, z_0 > 0$. In the first case, we have a decaying Gamow vector. Its form is given by [79,80]:

$$\chi^{\text{decaying}}(r;z_0) = \begin{cases} \sin(k_0 r), & 0 < r < a \ , \\ \alpha_2(k_0)\mathrm{e}^{\mathrm{i}Q_0 r} + \beta_2(k_0)\mathrm{e}^{-\mathrm{i}Q_0 r}, & a < r < b \ , \\ \mathscr{F}_1(k_0)\mathrm{e}^{\mathrm{i}k_0 r}, & b < r < \infty \ , \end{cases} \tag{B.22}$$

where k_0 and Q_0 are as in (B.17). If instead of z_0, we have its complex conjugate z_0^*, then $\mathrm{Im}\, z_0^* > 0$ and we have the growing Gamow vector $\chi^{\text{growing}}(r;z_0^*)$. To obtain its explicit form, we replace z_0 by z_0^* in (B.22).

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