Schematic model for QCD. I. Low energy meson states

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A simple model for QCD is presented, which is able to reproduce the meson spectrum at low energy. The model is a Lipkin-type model for quarks coupled to gluons. The basic building blocks are quark-antiquark pairs coupled to a definite flavor and spin. These pairs are coupled to pairs of gluons with spin zero. The multiplicity problem, which dictates that a given experimental state can be described in various manners, is removed when a particle-mixing interaction is turned on. In this first paper of a series we concentrate on the discussion of meson states at low energy, the so-called zero temperature limit of the theory. The treatment of baryonic states is indicated, also.

I. INTRODUCTION

QCD is the favored theory of the strong interactions. At low energy, however, the description of the hadronic spectrum based on QCD becomes difficult due to the nonlinear structure of the theory. This nonperturbative regimen, in contrast to lattice gauge calculations, can be explored by means of schematic models. The use of such models is common in other fields of physics where the many-body structure of the theory may be explored by introducing effective degrees of freedom and their couplings. Nuclear structure physics is one of these examples and it is as complicated and involved as the low-energy domain of QCD. Like in the case of nuclear structure physics, QCD descriptions based on simple models may help in the understanding of basic concepts and procedures. The Lipkin model [1] is one of the most famous schematic models, and it helped substantially to appreciate the importance of pairing two-body interactions as well as the importance of collectivity in building the low-energy part of the nuclear spectrum. An extended version of the Lipkin model was applied to the description of pion condensates in nuclei [2]. A variety of many body techniques have been tested with Lipkin-type models [3,4]. In Ref. [3] some realistic, less schematic, nuclear interactions, suitable to describe various nuclear properties, were investigated in this way. In Ref. [5] a Lipkin model was applied to describe a system of many quarks. As seen in these examples, the predictions of schematic models can be also rather rich in their complexities. This fact was shown, for a simple model of many gluon systems, in Ref. [6].

Until now, the only formalism that can handle QCD from first principles is the lattice gauge theory [7]. Particularly, in many gluon systems, a good description is obtained without considering finite volume effects [8]. The problem with lat-

tice gauge calculations, to treat QCD at low energies, is that only the lowest state, and in some cases also the next to the lowest state, for a given spin, charge conjugation, and parity, can be calculated. Lattice theory is numerically quite involved, and the inclusion of quarks and antiquarks brings in additional problems that are difficult to solve. Effective models of the hadrons, such as the MIT model [9], can help to shed some light on the structure of QCD at low energy. In Ref. [6] the spectrum of gluons, as obtained in Ref. [8], was reproduced and the sequence of levels explained by simple assumptions. In other works [10,11] many-body methods were used to describe the spectrum of QCD at low energy. After these considerations, it is obvious that it would be nice to have a model that (i) must be able to describe the basic structure of QCD at low and high energies, and (ii) must be solvable exactly. Probably, such a model does not exist, due to the complicated structure of QCD. Nevertheless, one can try to construct a model that comes as near as possible to QCD.

The purpose of the present work is to present a model that fulfills the above requirements. In Ref. [12] the most simple version of such a model was presented. Like the models mentioned at the beginning, it is based on a Lipkin-type model and consists of two levels for the description of quarks and antiquarks. These quarks are coupled to a boson level that describes gluon pairs coupled to spin zero. The other gluon states are treated as spectators. The basic ingredients of the model are the quark-antiquark pairs coupled to flavor singlet and spin zero and gluon pairs with spin zero. In Ref. [12] it is shown that the model is able to describe the appearance of a quantum phase transition at zero temperature, when the interaction is turned on, and a phase transition to the noninteracting case at nonvanishing temperature. In Ref. [12] the basic features of the model were discussed. Only flavor singlet and spin zero mesons were taken into account. The appearance of a Goldstone boson was obtained for a sufficiently strong interaction. This state consisted of a meson with negative parity. Also, it possesses a very collective nature, i.e., it is a superposition of many particles' (quarks, antiquarks, and gluons) states. The behavior of the model at high temperature was discussed, together with some

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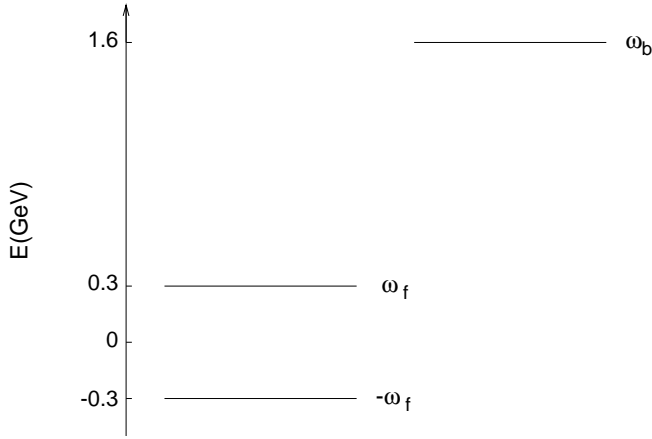‖Email address: reboiro@fisica.unlp.edu.ar

FIG. 1. Schematic representation of the model space. The fermion levels are indicated by their energies $\pm\omega_f$. The gluon pairs are represented by the level at the energy ω_b.

consequences for the quark-gluon plasma (QGP) in Refs. [13,14].

In the first part of the paper we will introduce the general form of the model for the description of meson states. The discussion will concentrate on the behavior of the model at low energy, corresponding to the zero temperature regime of QCD. The study of the high energy behavior and the transition to the quark gluon plasma [13,14] will be presented in the forthcoming paper of the series [15]. In Sec. II the basic ingredients of the model are introduced, with the proposition of a Hamilton operator. Because of the difficulties in treating fermion pairs exactly, we shall diagonalize them in a boson mapping scheme. The basis used to deal with the bosonic images of the fermion pairs, and the corresponding matrix elements of the proposed Hamiltonian, is also given in Sec. II. There we show how to assign charge conjugation and G-parity symmetries to the states belonging to the basis. In Sec. III the model is applied to the description of the low energy meson spectrum. Conclusions are drawn in Sec. IV.

II. THE MODEL

As indicated in Ref. [12], the fermion sector is described by a Lipkin-type model [1], consisting of two levels, one at energy $-\omega_f$ and the other at energy $+\omega_f$ (see Fig. 1). This is the Dirac picture for fermions, where antiquarks are regarded as holes in the lower level. Alternatively, one also describes quarks and antiquarks in the level at $+\omega_f$. The quarks and antiquarks are coupled to a 1.6-GeV level, which is occupied by gluon pairs with spin zero [6], a value that was obtained in the description of a many gluon system and which was adjusted to lattice gauge calculations [8]. In consequence, we shall take the energy of the gluon pair state as externally fixed data. The value ω_f is fixed to one third of the nucleon mass (0.33 GeV). There are further gluon states [6] that do not interact with the quarks and antiquarks. These states will be treated as spectators and should be taken into account in the final spectrum.

The degeneracy of each fermion level is 2Ω, where Ω

refers to color, flavor, and eventually other degrees of freedom. If only spin (n_s), flavor (n_f), and color (n_c) degrees of freedom are considered, the value of Ω is given by the product $2\Omega = n_c n_f n_s$. For flavor (0,0) and spin 0 pairs, only, the model has similarities to the one of Ref. [2], but with a different interaction. In Ref. [2] the pion condensate in nuclei was the dominant phenomenon. In the present case, nucleons are replaced by quarks and the pions by gluons. The model has some similarities to Ref. [5], which is also a Lipkin-type model. There, only quarks were considered and the interaction conserves their number.

For zero temperature and no interactions the lower level is filled by fermions. The creation (annihilation) operators of these fermions are $c^\dagger_{\alpha(1,0)f\sigma i}$ ($c^{\alpha(1,0)f\sigma i}$), in covariant and contravariant notation for the indices. The symbol $(1,0)f$ refers to the flavor part, where $(1,0)$ is the SU(3)-flavor notation and f is a shorthand notation for the hypercharge Y, the isospin T, and its third component T_z. The index σ represents the two spin components $\pm\frac{1}{2}$, the index $i=1$ or 2 stands for the upper or lower level, and the index α represents all remaining degrees of freedom, which are at least 3 because of the color degree of freedom (when only color is taken into account, instead of α we will use the index c). Lowering and raising the indices of the operators introduces a phase, which depends on the convention used [16], and a change of the indices to their conjugate values, i.e., the quantum numbers $(1,0)YTT_z\sigma$ change to $(0,1)-YT-T_z-\sigma$.

The operators, defined above, contain the relevant degrees of freedom of QCD, i.e., color, spin, and flavor. These basic degrees of freedom appear at all energies, no matter how the resulting particles are defined, i.e., either in the perturbative or in the nonperturbative regime. In the nonperturbative regime one usually denotes them as *constituent* or *effective* particles. This is mainly due to the difference in the spatial properties, while color, spin, and flavor have the same meaning as in QCD. Here, the quarks and antiquarks are *constituent* particles at *low energy* and have little in common (except for the quantum numbers mentioned) with the ones at high energy. We shall show that a model that contains these basic degrees of freedom *and* which takes into account the dynamic coupling with gluons can describe the main characteristics of QCD at low energy.

The quark and antiquark creation and annihilation operators are given in terms of the operators c and $c^\dagger$,

$$a^\dagger_{\alpha f\sigma} = c^\dagger_{\alpha f\sigma 1}, \quad d_{\alpha f\sigma} = c^\dagger_{\alpha f\sigma 2},$$

$$a^{\alpha f\sigma} = c^{\alpha f\sigma 1}, \quad d^{\dagger\alpha f\sigma} = c^{\alpha f\sigma 2}, \tag{1}$$

which corresponds to the Dirac picture of particles and antiparticles: quarks are described by fermions in the upper level and antiquarks by holes in the lower level.

The gluon sector of the model space is described by bosons that represent pairs of gluons coupled to spin zero. The energy of a boson state is fixed at the value $\omega_b = 1.6$ GeV [6], as mentioned before.

The quark-antiquark pairs of the model are given by

$$C^{f_2\sigma_2 2}_{f_1\sigma_1 1}=B^{\dagger f_2\sigma_2}_{f_1\sigma_1}=\sum_\alpha c^\dagger_{\alpha f_1\sigma_1 1}c^{\alpha f_2\sigma_2 2}=\sum_\alpha a^\dagger_{\alpha f_1\sigma_1}d^{\dagger\alpha f_2\sigma_2},$$

$$C^{f_2\sigma_2 1}_{f_1\sigma_1 2}=B^{f_2\sigma_2}_{f_1\sigma_1}=\sum_\alpha c^\dagger_{\alpha f_1\sigma_1 2}c^{\alpha f_2\sigma_2 1}=\sum_\alpha d_{\alpha f_1\sigma_1}a^{\alpha f_2\sigma_2},$$

$$C^{f_2\sigma_2 1}_{f_1\sigma_1 1}=\sum_\alpha c^\dagger_{\alpha f_1\sigma_1 1}c^{\alpha f_2\sigma_2 1}=\sum_\alpha a^\dagger_{\alpha f_1\sigma_1}a^{\alpha f_2\sigma_2},$$

$$C^{f_2\sigma_2 2}_{f_1\sigma_1 2}=\sum_\alpha c^\dagger_{\alpha f_1\sigma_1 2}c^{\alpha f_2\sigma_2 2}=\sum_\alpha d_{\alpha f_1\sigma_1}d^{\dagger\alpha f_2\sigma_2}. \qquad (2)$$

The first two equations describe the creation and annihilation of quark-antiquark pairs. The pairs can be coupled to definite flavor $(\lambda,\lambda)=(0,0)$ or $(1,1)$ and spin $S=0$ or 1. We shall write, in this coupling scheme, $B^\dagger_{(\lambda,\lambda)f,SM}$, where f is the flavor, S is the spin, and M is the spin projection. The operators $B_{(\lambda,\lambda)f,SM}$ annihilate the vacuum $|0\rangle$, which can be taken as the configuration where the lower state is completely filled and the upper one is empty. Note that the vacuum state is not uniquely defined [12]. All states, which contain only quarks in the upper level and where the lower level is completely filled (so that antiquarks are not activated), regardless of color, as, for example, the three quark baryon states, are annihilated by $B_{(\lambda,\lambda)f,SM}$. This property derives from the fact that the operators $B_{(\lambda,\lambda)f,SM}$ contain an antiquark annihilation operator that anticommutes with all the quark creation operators. Therefore, the Hilbert space of the model may be divided into sectors, each one with a different vacuum state having a given baryon number. The one with the baryon number zero is the real particle vacuum.

A. Group theory of the fermion part

From now on, we restrict to $2\Omega=n_s n_c n_f=18$ with $n_s=2$, $n_c=3$, and $n_f=3$, for the spin, color, and flavor degrees of freedom, respectively. The largest group, whose generators are $c^\dagger_{c_1 f_1\sigma_1 i}c^{c_2 f_2\sigma_2 j}$ ($c_i=1,2,3$; $f_i=1,2,3$; $\sigma_i=1,2$; and $i,j=1,2$), is the U(4Ω) group. One possible group chain for the classification of the states, which include the flavor [SU$_f(3)$] and the spin [SU$_S(2)$] groups, is given by

$$\begin{array}{ccc} [1^N] & [h]=[h_1 h_2 h_3] & [h^T] \\ \mathrm{U}(4\Omega) & \supset\mathrm{U}(\tfrac{\Omega}{3})\otimes & \mathrm{U}(12) \\ \cup & \cup & \\ (\lambda_C,\mu_C)\mathrm{SU}_C(3) & (\lambda_f,\mu_f)\mathrm{SU}_f(3)\otimes & \mathrm{SU}_S(2)S,M, \end{array} \qquad (3)$$

where the irreducible representation (irrep) of U(4Ω) is the completely antisymmetric one and N is the number of particles involved. The upper index in $[h^T]$ refers to the transposed Young diagram of $[h]$, where the columns and rows are interchanged [17]. Due to the antisymmetric irrep $[1^N]$ of U(4Ω), the irreps of U$(\Omega/3)$ and U(12) are complementary and the irrep of U$(\Omega/3)$, which is for $\Omega=9$ the color group,

has maximally three rows [17]. In group chain (3) no multiplicity labels are indicated. There is a multiplicity ρ_f for (λ_f,μ_f) and ρ_S for the spin S. The color labels (λ_C,μ_C) are related to the h_i via $\lambda_C=h_1-h_2$ and $\mu_C=h_2-h_3$. The complete state is given by

$$|N,(\lambda_C,\mu_C),\rho_f(\lambda_f,\mu_f)YTT_z,\rho_S SM\rangle, \qquad (4)$$

where Y is the hypercharge, T is the isospin and T_z its third component. For mesonlike states, the color quantum numbers to be considered are $(\lambda_C,\mu_C)=(0,0)$. These states will be located in an elementary volume of about $4\pi/3$ fm^3, corresponding to a sphere of radius 1 fm.

To obtain the values of h_i one has to consider all possible partitions of $N=h_1+h_2+h_3$, which fixes the color. For colorless states we have $h_1=h_2=h_3=h$. Each partition of N appears only once. The irrep $[hhh]$ of U$(\Omega/3)=$U(3) ($\Omega=9$) fixes the irrep of U(12), as indicated in Eq. (3). For the reduction of the irrep of U(12) we have written a computer code [18], which is available to the interested reader. As an example, let us consider the U(12) irrep $[3^6 0^6]$ and the two U(4) irreps $[9^2 0^2]$ and [9720], where the first one contains the state where the lower level is completely filled and the upper one empty, and the second irrep is the next highest one that contains flavor (0,0). The first is accompanied by flavor $(0,0)_1$ and the second one by $(0,0)_1$ and $(1,1)_1$, where the subindex denotes the multiplicity. The spin content of $[9^2 0^2]$ is given by 0_{55}, 1_{45}, 2_{36}, 3_{28}, 4_{21}, 5_{15}, 6_{10}, 7_6, 8_3, and 9_1. The spin content of [9720] is 0_{81}, 1_{171}, 2_{189}, 3_{135}, 4_{90}, 5_{54}, 6_{27}, and 7_9. The lowest dimensional irrep is $[5^2 4^2]$ with the spin content 0_1 and 1_1.

B. The boson mapping

The explicit construction of the basis states, Eq. (4), and the calculation of the matrix elements become very involved, which is in conflict with the idea to develop a simple model. A way out of it is to use a boson mapping of the pair operators $B^\dagger$ and B and work in the boson model space.

The basic ingredients of the model are the pair operators, given in Eq. (2). They can be mapped onto boson operators [19]

$$B^{\dagger f_2\sigma_2}_{f_1\sigma_1}\rightarrow b^{\dagger f_2\sigma_2}_{f_1\sigma_1}, \quad B^{f_2\sigma_2}_{f_1\sigma_1}\rightarrow b^{f_2\sigma_2}_{f_1\sigma_1}, \qquad (5)$$

where the operators on the right satisfy the normal boson commutation relations,

$$[b^{f_2\sigma_2}_{f_1\sigma_1},b^{\dagger f_4\sigma_4}_{f_3\sigma_3}]=\delta_{f_3 f_2}\delta_{f_4 f_1}\delta_{\sigma_3\sigma_2}\delta_{\sigma_4\sigma_1}. \qquad (6)$$

The exact boson mapping is quite involved, but it can be obtained in general [19–21]. For the sake of this work, it is worth to show that the mapping can be performed, indeed. We shall (i) work from the beginning in the boson space, (ii) define a Hamiltonian that corresponds to the Hamiltonian acting in the fermion space, and (iii) select a basis. The advantage of working in the boson space is the simplification in

getting the matrix elements (see below). The price to pay is related to the appearance of *nonphysical* states [19], as we shall discuss later on.

In order to choose a basis in the boson space, we profit from the fact that the basic degrees of freedom are given by the boson creation operators $b^\dagger_{(\lambda,\lambda)fSM} = b^\dagger_{\lambda fSM}$, with $\lambda = 0$ or 1 and $S = 0$ or 1. This gives four possible combinations of $[\lambda, S]$: [0,0], [0,1], [1,0], and [1,1]. Consequently, the total Hilbert space is the direct product of 1, 3, 8, and 24 dimensional harmonic oscillators [22].

For each harmonic oscillator we can define a seniority basis,

$$\mathcal{N}_{N_{\lambda S}\nu_{\lambda S}}(\boldsymbol{b}^\dagger_{\lambda S}\cdot\boldsymbol{b}^\dagger_{\lambda S})^{(N_{\lambda S}-\nu_{\lambda S})/2}|\nu_{\lambda S}\alpha_{\lambda S}\rangle, \qquad (7)$$

where $N_{\lambda S}$ is the number of bosons of type $[\lambda, S]$, $\nu_{\lambda S}$ the corresponding seniority, and $\mathcal{N}_{N_{\lambda S}\nu_{\lambda S}}$ is a normalization constant. The seniority is the number of $\boldsymbol{b}_{\lambda S}$ bosons *not coupled* into pairs. The $\alpha_{\lambda S}$ contain all other quantum numbers for a particular harmonic oscillator. The dot in the factor refers to the scalar product.

The choice of a seniority basis is particularly useful for the calculation of the matrix elements of the interaction, which contains expressions of the form $(\boldsymbol{b}^\dagger_{\lambda S}\cdot\boldsymbol{b}^\dagger_{\lambda S})$, $(\boldsymbol{b}_{\lambda S}\cdot\boldsymbol{b}_{\lambda S})$, and $(\boldsymbol{b}^\dagger_{\lambda S}\cdot\boldsymbol{b}_{\lambda S})$, where the latter is just the number operator of the bosons of the type $[\lambda, S]$. The exact structure of $|\nu_{\lambda S}\alpha_{\lambda S}\rangle$ is not needed, except for the knowledge of the quantum numbers $\alpha_{\lambda S}$.

For the one dimensional harmonic oscillator ([0,0]) the seniority can take the values 0 or 1. The state is of the form $(\boldsymbol{b}^\dagger_{00})^{N_{00}}|0\rangle = (\boldsymbol{b}^\dagger_{00}\cdot\boldsymbol{b}^\dagger_{00})^{(N_{00}-\nu_{00})/2}(\boldsymbol{b}^\dagger_{00})^{\nu_{00}}|0\rangle$. For the three dimensional harmonic oscillator the seniority is equal to the spin $S_{\lambda S}$. The explicit expression of the state is given in Ref. [22]. The eight dimensional oscillator contributes to flavor only and it is discussed in Appendix A. The 24 dimensional oscillator can be found in Ref. [6], where the color part in Ref. [6] has to be interpreted here as the flavor part. In Ref. [6] only singlet states are listed, but the procedure to obtain nonsinglet flavor states is outlined.

The parity of each state is given by $P = (-1)^N$, where $N = \Sigma_{\lambda,S}N_{\lambda S}$ is the total number of bosons. Each boson stems from a particle-antiparticle pair, which carries negative parity.

In order to obtain the property under charge conjugation, one has to apply the charge conjugation operator $\boldsymbol{C}$ to the pair creation operator $\boldsymbol{B}^\dagger_{(\lambda,\lambda)f,SM}$. The result is (see Appendix B)

$$\boldsymbol{C}\cdot\boldsymbol{B}^\dagger_{\lambda f,SM}\cdot\boldsymbol{C}^{-1} = (-1)^S\boldsymbol{B}^\dagger_{\bar\lambda\bar f,SM}, \qquad (8)$$

where $\bar\lambda = (\lambda,-\mu), \bar f = -Y, T, -T_z$. From this it is clear that only states with $Y = 0$, $T_z = 0$, and $\mu = 0$ can have a definite C parity. In Eq. (8) we make use of the application of the operator $\boldsymbol{C}$, which interchanges quark and antiquark opera-

tors $(a^\dagger\leftrightarrow d^\dagger)$ and inverts the magnetic quantum numbers $(Y_i\rightarrow -Y_i$ and $T_{iz}\rightarrow -T_{iz})$ of flavor and of color only.[1]

For products of two pair creation operator we obtain

$$\boldsymbol{C}[\boldsymbol{B}^\dagger_{\lambda_1,S_1}\otimes\boldsymbol{B}^\dagger_{\lambda_2,S_2}]^{\rho(\lambda,\mu)}_{f,M}\boldsymbol{C}^{-1}$$
$$= (-1)^{S_1+S_2-\lambda-\mu+\rho_{max}-\rho}[\boldsymbol{B}^\dagger_{\lambda_1,S_1}\otimes\boldsymbol{B}^\dagger_{\lambda_2,S_2}]^{\rho(\mu,\lambda)}_{\bar{\mathbf{f}},M}, \qquad (9)$$

where ρ is the multiplicity label of (λ,μ) in the product $(\lambda_1,\lambda_1)\otimes(\lambda_2,\lambda_2)$. The symbol ρ_{max} denotes the maximal value of ρ. The phase convention of Ref. [23] was used. The symbol $\otimes$ denotes the combined product in $SU_f(3)$ and $SU_S(2)$.

In analogy, the action of the charge conjugation on a product of three pair operators can be obtained:

$$\boldsymbol{C}\{[\boldsymbol{B}^\dagger_{\lambda_1,S_1}\otimes\boldsymbol{B}^\dagger_{\lambda_2,S_2}]^{\rho_{12}(\lambda_{12},\mu_{12}),S_{12}}\otimes\boldsymbol{B}^\dagger_{\lambda_3,S_3}\}^{\rho(\lambda,\mu),S}_{f,M}\boldsymbol{C}^{-1}$$
$$= (-1)^{S_1+S_2+S_3-\lambda-\mu+\rho_{12,max}-\rho_{12}+\rho_{max}-\rho}$$
$$\times\{[\boldsymbol{B}^\dagger_{\lambda_1,S_1}\otimes\boldsymbol{B}^\dagger_{\lambda_2,S_2}]^{\rho_{12}(\mu_{12},\lambda_{12}),S_{12}}\otimes\boldsymbol{B}^\dagger_{\lambda_3,S_3}\}^{\rho(\mu,\lambda),S}_{\bar{\mathbf{f}},M}, \qquad (10)$$

where ρ_{12} is the multiplicity of (λ_{12},μ_{12}) in the product of $(\lambda_1,\lambda_1)\otimes(\lambda_2,\lambda_2)$, ρ is the multiplicity of the total irrep in the last coupling of the above equation, and $\rho_{12,max}$ is the maximal value of ρ_{12}.

The procedure outlined here can be used in a recursive way for more involved coupling schemes. For our purpose it is sufficient to go up to three pairs, which will be the dominant structure at low energy.

For the G parity the additional rotation in the isospin space has to be applied, which changes $T_{i,z}$ to $-T_{i,z}$ [24]. For a polynomial in the pair operators this gives an additional phase $(-1)^T$, where T is the total isospin [24].

The same phase properties under C- and G-parity transformations have to be valid for the mapped boson operators $b^\dagger_{\lambda f,SM}$.

In a seniority basis, the matrix elements are easily obtained, and they are written as

$$\langle N_{\lambda S}+2\,\nu_{\lambda S}\alpha_{\lambda S}|(\boldsymbol{b}^\dagger_{\lambda S}\cdot\boldsymbol{b}^\dagger_{\lambda S})|N_{\lambda S}\nu_{\lambda S}\alpha_{\lambda S}\rangle$$
$$= \sqrt{(N_{\lambda S}-\nu_{\lambda S}+2)(N_{\lambda S}+\nu_{\lambda S}+d_{\lambda S})},$$

$$\langle N_{\lambda S}-2\,\nu_{\lambda S}\alpha_{\lambda S}|(\boldsymbol{b}_{\lambda S}\cdot\boldsymbol{b}_{\lambda S})|N_{\lambda S}\nu_{\lambda S}\alpha_{\lambda S}\rangle$$
$$= \sqrt{(N_{\lambda S}-\nu_{\lambda S})(N_{\lambda S}+\nu_{\lambda S}+d_{\lambda S}-2)},$$

$$\langle N_{\lambda S}\nu_{\lambda S}\alpha_{\lambda S}|(\boldsymbol{b}^\dagger_{\lambda S}\cdot\boldsymbol{b}_{\lambda S})|N_{\lambda S}\nu_{\lambda S}\alpha_{\lambda S}\rangle = N_{\lambda S}, \qquad (11)$$

where $d_{\lambda S}$ is 1, 3, 8, or 24 for the case of the 1, 3, 8, or 24 dimensional harmonic oscillator. As a shorthand notation we

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[1]Hereafter, for the sake of notation, we shall indicate charge conjugate states with a bar over the index f.

will use instead of $(\boldsymbol{b}^{\dagger}_{\lambda S} \cdot \boldsymbol{b}^{\dagger}_{\lambda S})$ the expression $(\boldsymbol{b}^{\dagger}_{\lambda S})^2$, and similarly for the other products, $(\boldsymbol{b}_{\lambda S})^2$ and $\boldsymbol{b}^{\dagger}_{\lambda S} \cdot \boldsymbol{b}_{\lambda S}$.

As a Hamiltonian, invariant under rotation, charge conjugation, and G parity, we propose

$$\boldsymbol{H} = 2\omega_f \boldsymbol{n}_f + \omega_b \mathbf{n_b} + \sum_{\lambda S} V_{\lambda S} \Bigg\{ [(\boldsymbol{b}^{\dagger}_{\lambda S})^2 + 2\boldsymbol{b}^{\dagger}_{\lambda S} \cdot \boldsymbol{b}_{\lambda S} + (\boldsymbol{b}_{\lambda S})^2]$$

$$\times \left(1 - \frac{\boldsymbol{n}_f}{2\Omega}\right) \boldsymbol{b} + \boldsymbol{b}^{\dagger} \left(1 - \frac{\boldsymbol{n}_f}{2\Omega}\right) [(\boldsymbol{b}^{\dagger}_{\lambda S})^2 + 2\boldsymbol{b}^{\dagger}_{\lambda S} \cdot \boldsymbol{b}_{\lambda S}$$

$$+ (\boldsymbol{b}_{\lambda S})^2] \Bigg\}. \tag{12}$$

Due to symmetry arguments, the interaction strength $V_{\lambda S}$ is the same for the two last lines in Eq. (12). The term $(\boldsymbol{b}^{\dagger}_{\lambda S})^2 [(\boldsymbol{b}_{\lambda S})^2]$ describes the creation (annihilation) of two quark-antiquark pairs with the simultaneous creation or annihilation of a gluon pair. The term $\boldsymbol{b}^{\dagger}_{\lambda S} \cdot \boldsymbol{b}_{\lambda S}$, in Eq. (12), describes the scattering of a fermion pair with the emission or annihilation of a gluon pair. All processes can be depicted by a Feynman graph and all graphs can be obtained from any other one by an appropriate interchange of lines. Because the strength $V_{\lambda S}$ should be, basically, invariant under the exchange of lines, we shall use the same interaction strength for all channels, as a first approximation. The terms that appear in Eq. (12) originate in the normal product of $:(\boldsymbol{b}^{\dagger}_{fS} + \boldsymbol{b}_{fS})^2:$, where the square implies a scalar product. The factor $(1 - \boldsymbol{n}_f/2\Omega)$ represents a cutoff that can be traced back to the boson mapping of the fermion pairs with flavor (0,0) and spin 0. This term simulates the effect of an exact boson mapping [19–21] and it is responsible for the disappearance of the interaction when the number of pairs reaches 2Ω. In other words, this cutoff term simulates the Pauli principle that does not allow more than 2Ω pairs.

Hamiltonian (12) is the most simple form we can think of and it contains only four parameters (the values of $V_{\lambda S}$). The value of ω_f is fixed to 0.33 GeV, which is about $\frac{1}{3}$ of the mass of a nucleon. The most notorious difficulty, associated with the use of the boson mapping, lies in the Hilbert space of the boson operators. It is larger than the Hilbert space of the fermion pairs. In some situations one can identify the source of the spurious dimensions. If, for example, only flavor (0,0) and spin 0 pairs are taken into account, the relevant group structure is $U(4\Omega) \supset U(2\Omega) \otimes U(2)$. The irrep of $U(4\Omega)$ has to be antisymmetric, which implies that the irreps of $U(2\Omega)$ and $U(2)$ have to be complementary, i.e., if $U(2)$ is given by a Young diagram with two rows, the one of $U(2\Omega)$ has to be the adjoint, which is obtained by interchanging rows and columns [17]. The upper limit, up to which no spurious states appear, is 2Ω because $U(2\Omega)$ allows 2Ω rows in the Young diagram. This is also the maximum number of pairs allowed, i.e., for this case no unphysical states occur. If flavor values (0,0) and (1,1) and spin 0 are used, only, we have $U(4\Omega) \supset U(2\Omega/3) \otimes U(3)$ and up to $2\Omega/3$ pairs there is no problem with respect to the appearance of unphysical states. This implies that states with explicit flavor will present unphysical states only for large

number of bosons. If flavor (0,0) and spins 0 and 1 are considered, we have $U(4\Omega) \supset U(\Omega) \otimes U(4)$, which has as an upper limit the number Ω up to which no spurious states appear. Finally, for all pairs, i.e., flavor (0,0), (1,1) and spin 0,1 the relevant group chain is $U(4\Omega) \supset U(\Omega/3) \otimes U(12)$ and the upper limit is $\Omega/3$. This gives us a hint about the group sequence where unphysical states may appear. The upper limit up to which all states are physical is lowered in the sequence where bosons with more degrees of freedom appear. There are states that can be described either by one or the other type of bosons or even by a combination of bosons. For example, when both levels, the lower and the upper ones, are filled there is only one allowed state, which is the flavor singlet with spin zero. However, all types of boson pairs can describe it, e.g., when the number of bosons coupled to flavor singlet and spin zero is equal to 2Ω. In view of these considerations, and for the 1, 3, 8, and 24 dimensional harmonic oscillator basis, we have introduced the limits 2Ω, Ω, $2\Omega/3$, and $\Omega/3$, respectively. The higher nonphysical states do not play an essential role because, as shown below, the dominant contribution at low energy comes from configurations with a small number of quark-antiquark pairs [15]. By working with these dimensional cutoff values, the influence of nonphysical states is minimized. Also, for each case, the total number of bosons is restricted to $\leq 2\Omega$. For a reasonable interaction strength, however, the dominant contribution comes from a small number of bosons. In such cases, the number of unphysical states is small and they do not influence much the result. The dimensional cutoff in the Hilbert space has the advantage that most unphysical states are excluded. In principle, one can eliminate the spurious states by applying another, more involved, procedure. For that one has to reduce the irrep of $U(12)$ to the flavor and spin groups, as done in the preceding section. This gives us the allowed content of flavor and spin for a given irrep of $U(12)$. The matching condition, i.e., by comparing for a given number of pairs the spin and flavor content on the boson side to the one on the fermion side, eliminates unphysical states. If on the boson and fermion sides, for a given flavor and spin, the number of states are equal, all states in the model space are taken into account. This is the case of low lying basis states. A simple counting procedure can be used for other situations. If there are, for a given flavor and spin, more states in the boson space than in the fermion space, one cannot decide easily which combination is allowed. However, one can reduce the number of states of the model space to the same dimension as the one of the fermion space. As a rule, one can first eliminate the states that contain most of the bosons with the largest degree of freedom, i.e., which are of the type [1,1], and in this way proceed, if necessary, until only states with flavor (0,0) and spin zero bosons are left. At least, the proposed procedure eliminates most of the spurious states. The error made can be absorbed in the parameters of the model, a general practice in dealing with phenomenological models, because in the end the correct number of degrees of freedom (dimension of the Hilbert space) dominates in a successful description of the spectrum. The idea of the proposed procedure is not new and it was used in another context by Cseh and Lévai [25].

TABLE I. States used in the fit. The particles are listed in the first column, and their transformation properties in flavor and spin are shown in the second and third columns. Note that, for the particles in the first (0,0), (1,1) 0^- and (0,0), (1,1) 1^- irreps, we are listing the value of the masses without flavor mixing (they are marked by an asterisk). The experimental data are taken from Ref. [27].

Particle	(λ_f, μ_f)	J^π	E_{th} (GeV)	$E_{\exp}$ (GeV)
Vacuum	(0,0)	0^+	0.0	0.0
f_0(400–1200)	(0,0)	0^+	0.656	0.600
f_0(980)	(1,1)	0^+	0.797	0.980
f_1(1420)	(0,0)	1^+	1.445	1.420
f_2(1270)	(1,1)	1^+	1.363	1.270
η'(958)	(0,0)	0^-	0.885	*0.892
η(1440)	(0,0)	0^-	1.379	1.440
η(541)	(1,1)	0^-	0.602	*0.615
η(1295)	(1,1)	0^-	1.428	*1.295
η(1760)	(1,1)	0^-	1.671	1.760
ω(782)	(0,0)	1^-	0.851	*0.861
ϕ(1020)	(1,1)	1^-	0.943	*0.940
ω(1420)	(1,1)	1^-	1.389	1.420
ω(1600)	(1,1)	1^-	1.639	1.650

III. APPLICATION TO THE MESON SPECTRUM

The Hamiltonian (12) commutes with the isospin operator and it does not depend explicitly on the hypercharge Y. As a consequence, all states that belong to the same flavor irrep are degenerate. In principle, we can add terms proportional to T^2, Y, and Y^2 in order to lift the degeneracy. These terms will add new parameters to the four already present (ω_b is fixed, as in Ref. [6]). Also, a flavor mixing term could be added, as suggested by the η-η' mixing [26]. In order to simplify the discussion, we shall first ignore these additional interaction terms.

In fitting the spectrum of the mesons we will use, as an experimental input for each multiplet, only the state with T $=0$ and $Y=0$. For an octet all states have the same energy as the isospin singlet and hypercharge zero state. Because later on we shall take into account flavor mixing interactions too, the positions of the singlet and octet states are not fixed at the measured energy values, but at the values obtained when the flavor mixing interactions are switched off [24]. The mixing angle is introduced for two multiplets: the (1,1) 0^- and (0,0) 0^- irreps, containing the pions and the η, η', and the (1,1) 1^-, (0,0) 1^-, containing the ω, ϕ, and ρ particles. The mixing angles are, respectively, $-23.7°$ and $35.3°$ [24]. For other multiplets one assumes that the mixing angle is zero, because of missing data, and because of the smallness of the energy splitting between members, as compared to the energy splitting within the multiplet that contains the pions or the ρ mesons. The uncorrected masses for, e.g., the octet (before flavor mixing) are $m_8 = 615$ MeV in the first case (see notation of Ref. [24]) and $m_8 = 940$ MeV for the second case (see also Table I).

In Table I we show the states used in the fit. Their flavor, spin, and parity are indicated together with the experimental values. In total, to perform the fit, we have considered thirteen states with spin zero and one in the four parameter fitting procedure. All other states are predicted, particularly those with spins 2 and 3.

In Fig. 2 we give the spectrum for spin 0 and 1 meson states *without* any particle number changing interaction. On each side of a level the flavor quantum numbers and its degeneracy are indicated. This serves to illustrate that the multiplicity at energies lower than 2 GeV is already very large. This is a consequence of the various manners in which the same set of quantum numbers can be obtained, for a given configuration, when many quarks, antiquarks, and gluons are considered. This is known as the *multiplicity problem*. The result of the best fit values, obtained after the interaction is turned on, is given in Figs. 3–6 for spins 0, 1, 2, and 3, respectively. Only states that correspond to nonexotic parity/charge conjugation quantum numbers are shown. Most of them appear above 2 GeV and some can be deduced from the gluon spectrum published in Ref. [6]. In Figs. 3–6 each
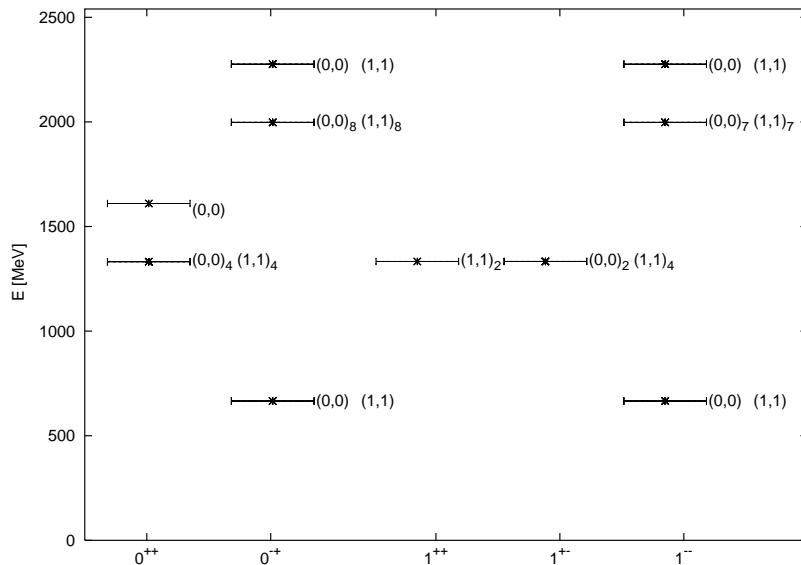

FIG. 2. The meson spectrum for spin 0 and 1 states, for the case of no interaction. The value $m_f = 0.33$ GeV was used. Note the large multiplicity appearing at already low energies.
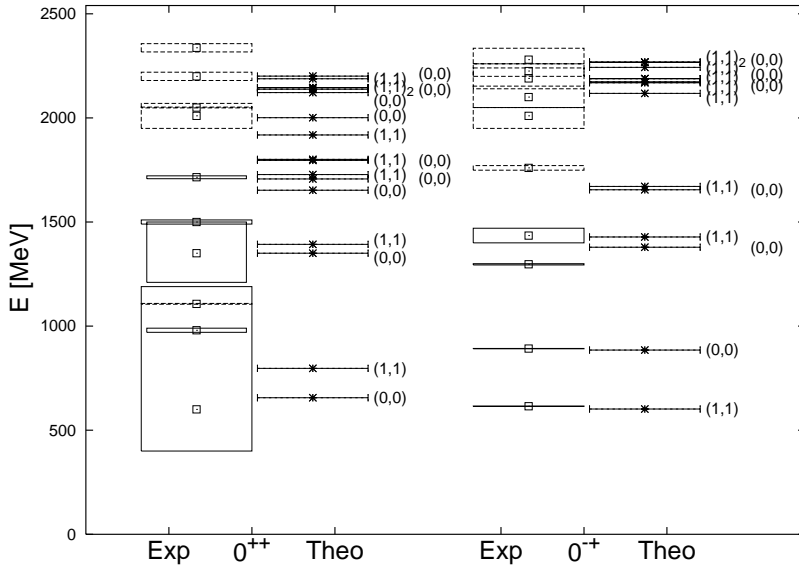
FIG. 3. The meson spectrum for spin 0, as obtained from the fit to experimental data [27].

theoretical spectrum is compared to the experimental one. On the right hand side of each level the theoretical interpretation in terms of flavor and the multiplicity is indicated. On the left hand side of each spin (J^{PC}) the experimental information is given, taken from Ref. [27]. The energy of these states, appearing in the summary table of Ref. [27], is given in boxes and the experimental error is reflected by the size of the box. If the error is very small, the box is replaced by a line. States that are not in the summary table of Ref. [27] are indicated by dashed boxes (lines). Only states that correspond to isospin singlet and hypercharge zero, after having corrected for the isospin mixing, are listed.

Note that very few states have a multiplicity. Most states were pushed upwards due to the interaction. This is an effect of the interaction, because it changes the number of particles and relates the quark-antiquark sector with the gluons. Models with a particle conserving interaction will always present the multiplicity problem. Thus, the particle mixing interaction is essential to remove the multiplicity problem.

The spin 2 and 3 states were not adjusted. The theoretical results seems to agree with the data, because the states are predicted at the correct energy domain. The density of states for a given flavor also seems to be reproduced.

In Table II we show the quark-antiquark pair and gluon pair contents of some selected states. The total number of quark-antiquark pairs is denoted by $\langle n_q \rangle$, where the symbol $\langle \cdots \rangle$ indicates the expectation value of this number. The quantity $\langle n_{ij} \rangle$ gives the average number of boson pairs of the type $[i,j]$, while $\langle n_g \rangle$ is the expectation value of the number of gluon pairs with spin zero. The total number of gluons is twice $\langle n_g \rangle$.

The structure of the calculated ground state (physical vacuum) is an interesting piece of information about the model's predictive power. The calculated value of the ground state energy is equal to -0.726 GeV. The physical vacuum state contains about 3.1 quark-antiquark pairs of type $[1,0]$, and the other configurations contribute with 0.06 pairs. The dominance of the $[1,0]$ quark-antiquark pairs is consistent
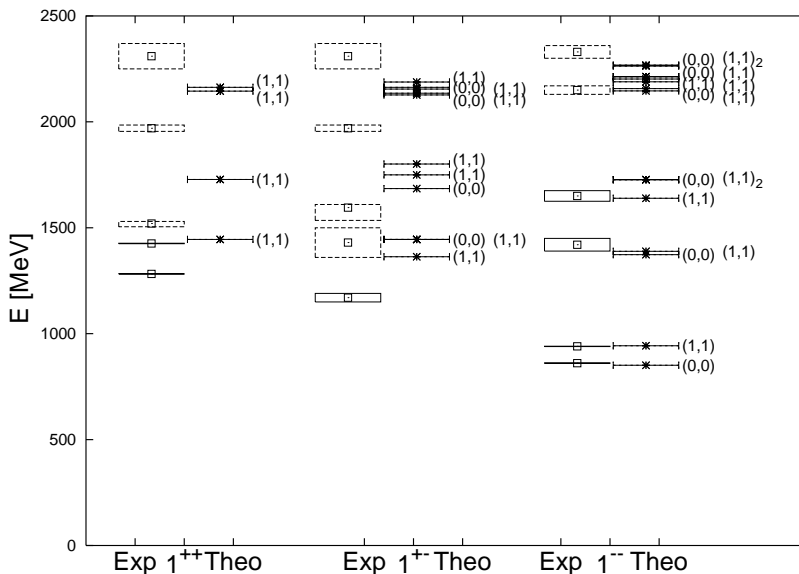

FIG. 4. The meson spectrum for spin 1, as obtained in a fit to experimental data [27].
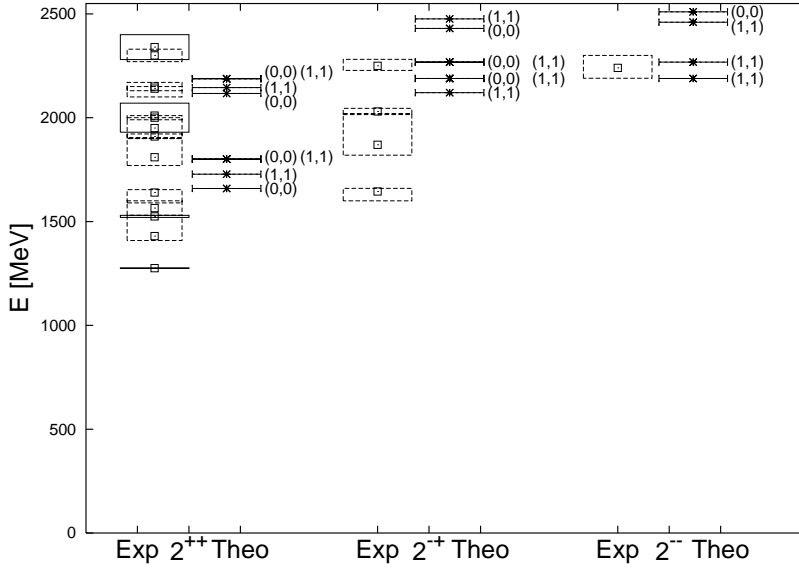
FIG. 5. The meson spectrum for spin 2, obtained with the parameters fixed by the fitting procedure. Experimental data are from Ref. [27].

with the strengths of the interactions. The parameters, obtained in the fit, are $m_f = 0.33$ GeV, $V_{0,0} = 0.0337$ GeV, $V_{0,1} = 0.0422$ GeV, $V_{1,0} = 0.1573$ GeV, and $V_{1,1} = 0.0177$ GeV. The channel [1,0] is clearly the strongest one. Also, the number of gluons is noticeable, i.e., 1.7 pairs, which correspond to more than three gluons contained in the elementary hadron volume. This indicates a collective behavior of QCD states at low energy. Indeed, the pion state [corresponding to $\eta(541)$ in Table II] contains about 2.7 pairs of the type [1,0] while the rest contributes with about 0.06 pairs, as in the ground state. The number of gluon pairs in the pion, 1.2 pairs or 2.4 gluons, is similar to the number of gluon pairs in the ground state. They constitute about 30% of the particle content. In brief, the calculated states contain a large number of quark-antiquark pairs and gluons. Roughly speaking, no single state can be described approximately by a pure quark-antiquark pair. Note that in this respect theory and experiment do agree, in spite of the simplicity of the model. We think that the complex structure of the meson

spectrum is described qualitatively by our model, as well as several other features, such as the positions of the first states with spins 2 and 3, the density of states with flavor (0,0) or (1,1) for each spin and parity, and charge conjugation number.

Concerning baryons, we have to include them yet in the model. Up to now they are described as spectators, i.e., without having an explicit coupling to the quark, antiquark, and gluon sea. For that, further interaction terms should be introduced. For example, one can introduce the interaction

$$\boldsymbol{n}_{D,(0,1)0}(\boldsymbol{b}^\dagger + \boldsymbol{b}) + \boldsymbol{n}_{D,(2,0)1}(\boldsymbol{b}^\dagger + \boldsymbol{b}), \qquad (13)$$

where $\boldsymbol{n}_{D,(\lambda,\mu)S}$ is the number operator of a *diquark* coupled to flavor (λ,μ) and spin S. This is analogous to the above *ansatz* of the Hamiltonian. The product of two pair creation operators of diquarks cannot appear because this would mix the baryon number. The interaction in Eq. (13) is a direct extension from the pair operator interaction of the former
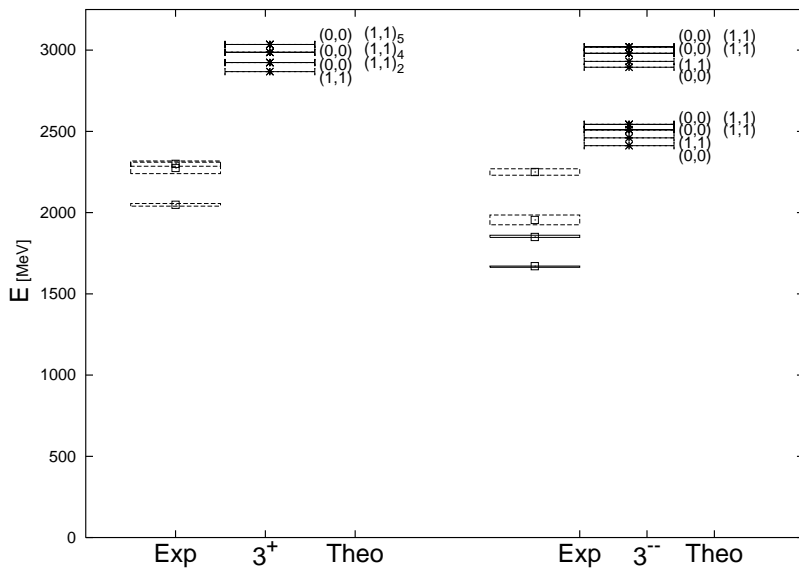

FIG. 6. The meson spectrum for spin 3. Experimental data are from Ref. [27].

TABLE II. Particle content for selected states. In columns we indicate the theoretical energy (E_{theo}), the flavor [(λ_f, μ_f)], spin J, and parity (π), expectation value of the boson pairs in the channel $(1,1)$ $0^-(\langle n_{10} \rangle)$, expectation value of the total number of quark-antiquark pairs ($\langle n_q \rangle$), and the total number of gluon pairs ($\langle n_g \rangle$) with spin 0.

Particle	E_{theo}	(λ_f, μ_f)	J^π	$\langle n_{10} \rangle$	$\langle n_q \rangle$	$\langle n_g \rangle$
Vacuum	0.0	(0,0)	0^+	3.118	3.177	1.705
$f_0(400-1200)$	0.656	(0,0)	0^+	0.457	0.471	0.321
$f_0(980)$	0.797	(1,1)	0^+	3.781	3.832	1.495
$f_1(1420)$	1.445	(0,0)	1^+	2.392	3.434	0.902
$f_2(1270)$	1.363	(1,1)	1^+	2.464	3.519	0.993
$\eta'(958)$	0.885	(0,0)	0^-	2.509	3.562	1.292
$\eta(1440)$	1.379	(0,0)	0^-	0.773	1.790	0.444
$\eta(541)$	0.602	(1,1)	0^-	2.711	2.766	1.163
$\eta(1295)$	1.428	(1,1)	0^-	1.611	1.638	0.531
$\eta(1760)$	1.671	(1,1)	0^-	3.535	4.581	1.254
$\omega(782)$	0.851	(0,0)	1^-	2.563	3.621	1.341
$\phi(1020)$	0.943	(1,1)	1^-	2.394	3.438	1.198
$\omega(1420)$	1.389	(1,1)	1^-	0.853	1.870	0.468
$\omega(1600)$	1.639	(1,1)	1^-	3.546	4.597	1.206

Hamiltonian. The first term in Eq. (13) acts only on states like the nucleon octet and the last one on particles like the baryon decouplet. The residual interaction will mix the number of gluons, and, mainly, increase the gluon content in the baryons, from a gluon content of about 30% to, say, 50%. The inclusion of baryons will be reported in another work [28] .

IV. CONCLUSIONS

In the present paper we have advanced a schematic model of QCD, based on a Lipkin-type model for fermions and an interaction to one gluon pair states. We have discussed the low energy structure of the model. The Hamiltonian is composed of a diagonal, particle conserving part, and an interaction that couples the quark-antiquark pairs to the gluons and changes the number of particles. The model contains only four parameters that were adjusted to reproduce 13 observed meson states with spins 0 and 1. After fixing these parameters, we have predicted the remaining part of the spectrum. The complex structure of the meson spectrum was qualitatively reproduced by our results.

Due to the schematic nature of the model, one cannot expect to be able to reproduce all details of the low energy meson spectrum. However, the results are in qualitative agreement with data, a fact that shows the validity of the model as a toy model for QCD. Baryons were not considered, but the extension to this sector was briefly indicated. The baryons would correspond to states where three extra quarks are added in the upper level. The corresponding operators will commute with the boson pair operators and an interaction of baryon states with the quark-antiquark sea should be included in the model.

We have found that the inclusion of particle mixing interactions turns out to be essential in order to remove the multiplicity problem encountered in other models, when states with many quarks and antiquarks are considered. This particle changing interaction also introduces ground state correlations resulting in many quarks, antiquarks, and gluon configurations in the states. It produces a large contribution of the gluons and the total spin is not a simple product of a quark-antiquark state, but of many quarks, antiquarks, and gluons. This illustrates the fact that, even at low energy, the structure of the hadron states is by no means as simple as suggested by earlier particle conservation models. It also shows that phenomenological potentials, which simulate the presence of gluons in a pure quark model, cannot resolve the problem of multiplicity.

APPENDIX A: THE EIGHT DIMENSIONAL OSCILLATOR

The reduction of the eight dimensional oscillator group, U(8), to the flavor group $SU_f(3)$ is discussed in Ref. [6]. As an intermediary group, between U(8) and $SU_f(3)$, one can use the SO(8) group. Though in Ref. [6] only the reduction to flavor singlet groups is listed, the general procedure is outlined. Programs are available on request [29] and the procedure has been published elsewhere [30].

The generators of the U(8) group are given by $b^\dagger_{(1,1)f_1,00} \cdot b_{(1,1)f_2,00}$, where the zeros refer to zero spin and its projection. Therefore, these bosons can only contribute to the flavor content. In Table III we list the flavor content of up to four bosons of the type $b^\dagger_{(1,1)f,00}$. As one can see, the multiplicity raises especially for the (1,1) flavor irrep. With the help of the $SO(8)$ group one can further reduce the multiplicity. For our purpose this is not necessary.

APPENDIX B: PARITY, CHARGE CONJUGATION, AND G PARITY

The charge conjugation operator acts as follows on the quark and antiquark creation operators:

$$C \cdot a^\dagger_{cf\sigma} \cdot C^{-1} = d^\dagger_{\bar{c}\bar{f}\sigma}, \quad C \cdot d^\dagger_{cf\sigma} \cdot C^{-1} = a^\dagger_{\bar{c}\bar{f}\sigma}, \quad \text{(B1)}$$

where the $a^\dagger$ transforms in color and flavor as a (1,0) SU(3) irrep, while the $d^\dagger$ transform as (0,1). If one of the color or flavor index is raised then the $d^\dagger$ transform as (1,0). The "$\bar{c}$" and "$\bar{f}$" refer to the reflection in the magnetic quantum numbers of SU(3). That is, f stands for Y, T, and T_z and $\bar{f}$ for $-Y$, T, and $-T_z$ and similarly for "$\bar{c}$."

With this, the action of the charge conjugation operator on a quark-antiquark pair is given by

TABLE III. The first column gives the Young diagram of the U(8) group, the second column the irrep of the flavor group $SU_f(3)$ and the third column gives the multiplicity of the flavor irrep. Only the flavor irreps (0,0) and (1,1) are listed.

U(8)	$SU_f(3)$	Multiplicity
[2]	(0,0)	1
	(1,1)	1
[1²]	(0,0)	0
	(1,1)	1
[3]	(0,0)	1
	(1,1)	1
[21]	(0,0)	0
	(1,1)	3
[1³]	(0,0)	1
	(1,1)	1
[4]	(0,0)	1
	(1,1)	2
[31]	(0,0)	0
	(1,1)	1
[2²]	(0,0)	2
	(1,1)	2
[21²]	(0,0)	1
	(1,1)	4
[1⁴]	(0,0)	1
	(1,1)	1

$$C \cdot B^\dagger_{\lambda f,SM} \cdot C^{-1} = C \sum_{cf_1f_2\sigma_1\sigma_2} a^\dagger_{cf_1\sigma_1} d^{\dagger c}_{f_2\sigma_2} \langle (1,0)f_1,(0,1)f_2|(\lambda_1,\lambda_1)f \rangle_1 \left(\frac{1}{2}\sigma_1,\frac{1}{2}\sigma_2 \Big| SM \right) C^{-1}$$

$$= \sum_{cf_1f_2\sigma_1\sigma_2} d^\dagger_{\bar{c}\bar{f_1}\sigma_1} \cdot a^{\dagger \bar{c}}_{\bar{f_2}\sigma_2} \langle (1,0)f_1,(0,1)f_2|(\lambda_1,\lambda_1)f \rangle_1 \left(\frac{1}{2}\sigma_1,\frac{1}{2}\sigma_2 \Big| SM \right)$$

$$= -\sum_{cf_1f_2\sigma_1\sigma_2} a^{\dagger c}_{\bar{f_2}\sigma_2} \cdot d^\dagger_{c\bar{f_1}\sigma_1} \langle (1,0)\bar{f_1},(0,1)\bar{f_2}|(\lambda_1,\lambda_1)\bar{f} \rangle_1 \left(\frac{1}{2}\sigma_1,\frac{1}{2}\sigma_2 \Big| SM \right)$$

$$= -(-1)^{2\lambda_1+1-S} \sum_{cf_1f_2\sigma_1\sigma_2} a^{\dagger c}_{\bar{f_2}\sigma_2} \cdot d^\dagger_{c\bar{f_1}\sigma_1} \langle (1,0)\bar{f_2},(0,1)f_1|(\lambda_1,\lambda_1)\bar{f} \rangle_1 \left(\frac{1}{2}\sigma_1,\frac{1}{2}\sigma_2 \Big| SM \right) = (-1)^S B^\dagger_{\lambda\bar{f},S_1M}, \quad \text{(B2)}$$

where we made use of the properties of the SU(2) [31] and SU(3) [23] Clebsch-Gordan coefficients. The subindex 1 in the SU(3) Clebsch-Gordan coefficient indicates a multiplicity of 1 [23].

For the product of two pair operators we have

$$C[B^\dagger_{\lambda_1,S_1} \otimes B^\dagger_{\lambda_2,S_2}]^{\rho(\lambda,\mu)}_{f,M} C^{-1} = C \sum_{f_1f_2M_1M_2} B^\dagger_{\lambda_1f_1,S_1M_1} \cdot B^\dagger_{\lambda_2f_2,S_2M_2} \langle (\lambda_1,\lambda_1)f_1,(\lambda_2,\lambda_2)f_2|(\lambda,\mu)f \rangle_\rho (S_1M_1,S_2M_2|SM) C^{-1}$$

$$= \sum_{f_1f_2M_1M_2} (-1)^{S_1+S_2} B^\dagger_{\lambda_1\bar{f_1},S_1M_1} \cdot B^\dagger_{\lambda_2\bar{f_2},S_2M_2} \langle (\lambda_1,\lambda_1)f_1,(\lambda_2,\lambda_2)f_2|(\lambda,\mu)f \rangle_\rho (S_1M_1,S_2M_2|SM)$$

$$= (-1)^{S_1+S_2-\lambda-\mu+\rho_{max}-\rho} \sum_{f_1f_2M_1M_2} B^\dagger_{\lambda_1\bar{f_1},S_1M_1} \cdot B^\dagger_{\lambda_2\bar{f_2},S_2M_2} \langle (\lambda_1,\lambda_1)\bar{f_1},(\lambda_2,\lambda_2)\bar{f_2}|(\lambda,\mu)\bar{f} \rangle_\rho$$

$$\times (S_1M_1,S_2M_2|SM)$$

$$= (-1)^{S_1+S_2-\lambda-\mu+\rho_{max}-\rho} [B^\dagger_{\lambda_1,S_1} \otimes B^\dagger_{\lambda_2,S_2}]^{(\mu,\lambda)}_{\bar{f},M}, \quad \text{(B3)}$$

where ρ is the multiplicity index of (λ,μ) in the product $(\lambda_1,\lambda_1)\otimes(\lambda_2,\lambda_2)$.

For the product of three pairs we have

$$
C\{[B^\dagger_{\lambda_1,S_1}\otimes B^\dagger_{\lambda_2,S_2}]^{(\lambda_{12},\mu_{12}),S_{12}}\otimes B^\dagger_{\lambda_3,S_3}\}^{(\lambda,\mu),S}_{f,M}C^{-1}
$$

$$
=\sum_{f_{12}f_3M_{12}M_3}C[B^\dagger_{\lambda_1 S_1}\otimes B^\dagger_{\lambda_2 S_1}]^{(\lambda_{12},\mu_{12})S_{12}}_{f_{12}M_{12}}C^{-1}
$$
$$
\times C\cdot B^\dagger_{\lambda_3 f_3 S_3 M_3}\cdot C^{-1}\langle(\lambda_{12},\mu_{12})f_{12},(\lambda_3,\lambda_3)f_3|(\lambda,\mu)f\rangle_\rho(S_{12}M_{12},S_3M_3|SM)C^{-1}
$$

$$
=\sum_{f_{12}f_3M_{12}M_3}(-1)^{S_1+S_2-\lambda_{12}-\mu_{12}+\rho_{12,max}-\rho_{12}}[B^\dagger_{\lambda_1 S_1}\otimes B^\dagger_{\lambda_2 S_1}]^{(\mu_{12},\lambda_{12})S_{12}}_{\bar{f}_{12}M_{12}}
$$
$$
\times(-1)^{S_3}B^\dagger_{\lambda_3\bar{f}_3 S_3 M_3}\langle(\lambda_{12},\mu_{12})f_{12},(\lambda_3,\lambda_3)f_3|(\lambda,\mu)f\rangle_\rho(S_{12}M_{12},S_3M_3|SM)C^{-1}
$$

$$
=(-1)^{S_1+S_2+S_3-\lambda_{12}-\mu_{12}+\rho_{12,max}-\rho_{12}}\sum_{f_{12}f_3M_{12}M_3}[B^\dagger_{\lambda_1 S_1}\otimes B^\dagger_{\lambda_2 S_1}]^{(\mu_{12},\lambda_{12})S_{12}}_{\bar{f}_{12}M_{12}}
$$
$$
\times B^\dagger_{\lambda_3\bar{f}_3 S_3 M_3}\langle(\mu_{12},\lambda_{12})\bar{f}_{12},(\lambda_3,\lambda_3)\bar{f}_3|(\mu,\lambda)\bar{f}\rangle_\rho(-1)^{\lambda_{12}+\mu_{12}-\lambda-\mu+\rho_{max}-\rho}(S_{12}M_{12},S_3M_3|SM)
$$

$$
=(-1)^{S_1+S_2+S_3-\lambda-\mu+\rho_{12,max}-\rho_{12}+\rho_{max}-\rho}\{[B^\dagger_{\lambda_1,S_1}\otimes B^\dagger_{\lambda_2,S_2}]^{(\mu_{12},\lambda_{12}),S_{12}}\otimes B^\dagger_{\lambda_3,S_3}\}^{(\mu,\lambda),S}_{\bar{\mathbf{f}},M}, \tag{B4}
$$

with the use of the notation of Ref. [23] for the SU(3) Clebsch-Gordan coefficients and their symmetry properties.

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