Collective and boson mapping description of a system of *N* Josephson junctions in a resonant cavity

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A system of *N* two-level Josephson junctions, interacting between themselves and with a single-mode cavity field, is described in terms of the superposition of fermionic and bosonic excitations. The results of the exact diagonalization are compared with the results of the Tamm-Dancoff approximation and with the results of a boson mapping. It is found that the boson mapping provides a suitable description of the spectrum, sum rules, and response function of the system. The dependence of the results upon the number of junctions, the excitation of the cavity modes, and the coupling strengths is investigated.

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I. INTRODUCTION

The physics of one-dimensional arrays of small Josephson junctions in a resonant cavity has been recently studied in a series of papers by Al-Saidi and Stroud[1,2] (see Ref. 3 for a two-dimensional array model). The approach followed by the authors of these works consisted of a description of the coupling between states of the junctions and the cavity mode as strongly entangled quantum states. Al-Saidi and Stroud have shown[1] that the spectrum corresponding to the one-photon process, obtained from an exact diagonalization in the space of the combined junction $\oplus$ photon states, agrees rather well with the spectrum obtained from a reduced Jaynes-Cummings Hamiltonian.[4] In Ref. 2 these authors have extended the model of Jaynes-Cummings to accommodate several junctions and cavity mode states, and they have outlined a comparison with the results of Dicke model.[5] Higher level effects caused by interactions between the junctions, which are not accounted for by the Dicke model, have been included perturbatively by adding a long-range dipole-dipole interaction term between the 1/2 pseudospins that represent each of the two-level junctions.[2]

Generally speaking, the models used so far have considered (a) interactions between junctions, and (b) interactions between junctions and cavity modes. When the atomic states in a junction are approximated as a two-level model situation,[4,5] the problem of placing several small Josephson junctions in a resonant cavity strongly resembles problems which involved fermion-boson interactions in other branches of physics.[6]

In the present work, we study the structure of the solutions of a Hamiltonian belonging to the class of models presented in Refs. 1,2,4,5 by applying techniques which originate in the nuclear many-body problem,[7] particularly in the treatment of the coupling between fermionic and bosonic degrees of freedom. Our starting point will be the description of a Josephson junction as a two-level system of fermions. While the cavity mode will play the role of a phonon, pairs of fermions will be represented as bosons.[8,9] The coupling between fermions and bosons will be treated microscopically by using a linearization procedure, which is the familiar Tamm-Dancoff approximation (TDA) of the nuclear many-body problem.[8] As an alternative method we advocate the use of a boson mapping technique.[9,10]

The details of the formalism are presented in Secs. II A–II D. The observables of the model are defined in Sec. II E. The results of our calculations are presented in Sec. III, where the results of the TDA and of the boson mapping technique are compared with the exact solution. We have taken the number of junctions, the frequency of the cavity mode, the strength of the coupling between the atomic excitations and the cavity mode, and the strength of the interaction between the junctions as parameters of the model. Conclusions are drawn in Sec. IV.

II. FORMALISM

The system under consideration, generally referred to as a one-dimensional array of *N* two-level Josephson junctions[1,2,4,11–14] will be considered as equivalent to a sample of *N* two-level atoms, each of them having two accessible atomic levels which may be excited or deexcited by either absorbing or emitting a photon such that the average distance between two atoms is much smaller than the wavelength of the radiation field (Dicke model).

The nonlinear phenomena arising in the interaction between optical photons and (effective) two-level atoms have been extensively treated in the literature (see, for instance,

Refs. 5,15–20). Explicitly, the Hamiltonian of such a physical system has the following contributions.

(a) The term which describes the free-photon field. This term is represented by a harmonic oscillator with a single frequency ω_b. The use of a single frequency is not a serious simplification of the problem, rather it is a convenient choice regarding the experimental situation.[21] In fact, one can consider a full spectrum, instead, without adding to the complexity of the problem.

(b) The term which describes the internal excitations of the atoms. The simplified form (e.g., a two-level model situation) allows for the use of su(2), pseudo-spin-1/2 operators and this part of the Hamiltonian has the form of a collective S_z term (the free-fermion field) plus an effective long-range spin-spin interaction.[22]

(c) The interaction between the photon field and the atoms. This is a scattering term representing the excitation and deexcitation of a single-atomic level by the absorption or emission of a photon. The scattering (or reemission) of the photon proceeds like a collective excitation, since it implies the participation of all the atoms in the system.

The exact treatment of the Hamiltonian consists of a diagonalization in the product space of photon and atomic states. The details of such a diagonalization can be found in Ref. 2. The obvious limitation of this treatment is posed by the dimensionality of the Hamiltonian matrix. Since the excitation and deexcitation of the atoms, by the multiple scattering of the photon field, may be considered as a collective phenomena (e.g., it implies the participation of all the atoms in the array), it opens the possibility to describe the physics of the problem from the point of view of current methods of the quantum many-body theory. The superposition of the multiple deexcitation and excitation of the atoms constitutes a sort of spin-up spin-down localized wave, of bosonic nature. Thus, the excitation of the atoms can be added to the external photon field in a nonperturbative way. In terms of the standard many-body theory, we may talk about a TDA state (e.g., the superposition of atomic excitations) built on the uncorrelated fermionic vacuum (all atoms in the lower-energy level). To this TDA state we add a bosonic excitation (the external photon field). Thus, the resulting spectrum may look as a nearly bosonic one. The departures from the truly bosonic behavior steam from the nature of the fermionic-pair contributions representing the excitation and deexcitation of the atomic energy levels. In this approximation, the number of atoms N plays a crucial role, since its inverse, $1/N$, has the meaning of an expansion parameter. Alternatively, one may think of an effective bosonic Hamiltonian. This construction may be obtained by applying another class of many-body techniques known as algebraic mappings.[10]

In the following sections we shall discuss the structure of the Hamiltonian[2] and the structure of the solutions which we have obtained by applying the above mentioned many-body techniques. We aim at the detailed comparison with the exact solution, obtained by a diagonalization, in order to support the approximations. In realistic cases (large values of N) the exact diagonalization is generally unfeasible and one may be forced to introduce approximations such as the many-body approximations which we are discussing in this paper.

A. The Hamiltonian

We shall consider a system of N Josephson junctions (atoms), each of them having two states. The energy gap between the states of a given junction is denoted by the quantity ω_f. The system of N junctions may interact with an external bosonic field of energy ω_b. A physical realization of this system would be the excitation of two-level atoms in a cavity by an incoming photon. We can write the Hamiltonian of the system as

$$H=\omega_f S_z+\omega_b(a^\dagger a+\tfrac{1}{2})+\zeta(a^\dagger S_-+S_+a)+\lambda\sum_{\substack{i,j=1\\ i\neq j}}^{N}(S_+^{(j)}S_-^{(i)}+S_+^{(i)}S_-^{(j)}), \tag{1}$$

where

$$S_+=\sum_{j=1}^{N}S_+^{(j)},$$

$$S_-=S_+^\dagger,$$

$$S_z=\sum_{j=1}^{N}S_z^{(j)} \tag{2}$$

are the collective ladder operators which raise (S_+), or lower (S_-) the states of the atoms, and S_z is the number operator for the pseudospin excitations. The second term of the Hamiltonian represents the free-photon field. The third term is the interaction of the photon with the atoms: the photon may be absorbed while the atoms are excited, or the photon may be emitted while the atoms are deexcited. The last term represents an effective atom-atom interaction that models the dynamical effects coming from the higher-energy levels of the junctions.[2] The collective operators S_+, S_-, and S_z obey the commutation rules of the su(2) algebra

$$[S_+,S_-]=2S_z,$$

$$[S_z,S_\pm]=\pm S_\pm. \tag{3}$$

The relationships between the coupling constants appearing in Eq. (1) and those of the original Hamiltonian of Refs. 1,2 are easily determined once the order of magnitude of the physical quantities of the system are fixed. The reader can verify the correspondence between the set ω_f, ω_b, ζ, λ and the central current, the activation energy, the capacitance, the frequency of the cavity mode, etc. (Ref. 28).

B. Exact solution

The operators $S_\pm^{(j)}$ and $S_z^{(j)}$ are the generators of the jth copy $\mathrm{su}(2)_j$ of the pseudospin algebra, where j is the atomic index. We take the tensor product $\Pi_{j=1}^{N}\mathrm{su}(2)_j$ as the carrier space for the representations of the fermionic part of the Hamiltonian. Afterwards, we consider the so-called Dicke state[5] as the collective state with $k\leqslant N$ atoms in the excited state

$$|k\rangle=\binom{N}{k}^{-1/2}\sum_{P}|k_1^P\cdots k_N^P\rangle,$$

$$|k_1^P\cdots k_N^P\rangle=N_{k^P}\prod_{j=1}^{N}S_+^{(j)k_j^P}|0\rangle,$$

$$k=\sum_{j=1}^{N}k_j^P,$$

$$\text{where}\quad k_j^P=0,1. \tag{4}$$

Note that the internal degeneracy of each of the two available atomic states is included in the definition of the basis $|k_1^P\cdots k_N^P\rangle$.

Since the Hamiltonian of Eq. (1), contains a bosonic degree of freedom, the state which represents l photons is written as the number state

$$|l\rangle=\frac{1}{\sqrt{l!}}a^{\dagger l}|0\rangle. \tag{5}$$

We shall then express the wave function of the photons and atoms as (dressed state)

$$|l,k\rangle=|l\rangle\otimes|k\rangle, \tag{6}$$

which is the direct product of bosonic and fermionic states. This is the basis where the Hamiltonian of Eq. (1) can be diagonalized exactly.

Since the Hamiltonian of Eq. (1) commutes with the operator

$$O=a^\dagger a+S_z+\tfrac{1}{2}N, \tag{7}$$

the basis of Eq. (5) may be labeled by the eigenvalues L of the operator O, namely,

$$O|l,k\rangle^{(L)}=L|l,k\rangle^{(L)}, \tag{8}$$

where $L=l+k$. Thus, the set $\{|l,k\rangle^{(L)}\}$ represents the states $|l,k\rangle$ to which the application of O assigns the same value of L. In the finite-dimensional subspace corresponding to a fixed value of L the matrix elements of the Hamiltonian read

$$\langle l,k|H|l,k\rangle^{(L)}=\omega_f(k-\tfrac{1}{2}N)+\omega_b(L-k+\tfrac{1}{2})+2\lambda k(N-k),$$

$$\langle l-1,k+1|H|l,k\rangle^{(L)}=\zeta\sqrt{(L-k)(k+1)(N-k)},$$

$$\langle l+1,k-1|H|l,k\rangle^{(L)}=\zeta\sqrt{(L-k+1)k(N-k+1)}. \tag{9}$$

C. Approximate solution: The TDA case

Approximate solutions of the Hamiltonian of Eq. (1) may be obtained by performing a linearization of the fermionic part of the Hamiltonian, namely,

$$H_f=\omega_f S_z+\lambda\sum_{\substack{i,j\\ i\neq j}}^{N}(S_+^{(j)}S_-^{(i)}+S_+^{(i)}S_-^{(j)}). \tag{10}$$

In the spirit of the TDA (Ref. 8) the linear combination

$$\gamma_n^\dagger=\sum_{j=1}^{N}X_{nj}S_+^{(j)} \tag{11}$$

is the operator which creates the nth superposition of the excitations of N atoms. Thus, the TDA approximation belongs to the class of boson expansions where the superposition of pairs of fermionic excitations, as the one induced by the action of the operators $S_+^{(j)}$ and $S_-^{(j)}$, is treated as a boson. Consequently, the fermionic Hamiltonian H_f acquires a diagonal form in the bosonic basis. Since the transformation between pair of fermions and bosons is an unitary transformation, each pair of fermions can be replaced by a superposition of boson and vice versa. The definition of the TDA bosons $\gamma_n^\dagger$ implies the existence of such an unitary transformation; which expresses the Hamiltonian, at leading order in some expansion parameter (which, as we shall show later on, is just the degeneracy of each atomic energy level) as an harmonic Hamiltonian and leaves the vacuum invariant. Then, we want to find the set of coefficients X_{nj} for each atom j and for each TDA mode n. The diagonalization of H_f in the basis $\gamma_n^\dagger$ yields a secular equation which is solvable and which determines the energy of the TDA modes W_n and the amplitudes X_{nj}. Therefore, if the dominant part of H_f is replaced by a superposition of the form $H_f=\text{const}+\Sigma_n W_n\gamma_n^\dagger\gamma_n$, the TDA linearization procedure of Ref. 8 yields the equation of motion

$$[H_f,\gamma_n^\dagger]=W_n\gamma_n^\dagger, \tag{12}$$

where W_n is the energy of the nth superposition. Solving Eq. (12) leads to the solutions of the amplitudes X_{nj} subject to the normalization condition

$$\langle[\gamma_n,\gamma_n^\dagger]\rangle=\sum_{j=1}^{N}|X_{nj}|^2 2\Omega_j=1. \tag{13}$$

The factor $2\Omega_j$ is the vacuum expectation value of the commutator between the operators $S_-^{(j)}$ and $S_+^{(j)}$ and it reads for the degeneracy of the levels of the jth atom. If we replace this value for 2Ω, adopting a common degeneracy for the atomic levels, the TDA eigenstates of the system are given by

$$W_1=\omega_f+(N-1)4\lambda\Omega$$

$$W_n=\omega_f-4\lambda\Omega,\quad n=2,3,\ldots,N. \tag{14}$$

The eigenstate with energy W_1 has the structure

$$\gamma_1^\dagger=\frac{1}{\sqrt{2\Omega N}}S_+, \tag{15}$$

which is the coherent superposition of all the atomic ladder operators $S_+^{(j)}$. Since the operators $\gamma_n^\dagger$ form a basis we can express the Hamiltonian of Eq. (10) in terms of the operators $\gamma_n^\dagger$ of Eq. (11). The TDA image of the Hamiltonian of Eq. (10) reads

$$H_f \rightarrow -\omega_f \Omega N + \sum_{m=1}^{N} W_m \gamma_m^\dagger \gamma_m . \tag{16}$$

The above results indicate that there is an state which is a coherent superposition of all the atomic excitations, the state $n=1$ or collective state, and $N-1$ states which are mostly related to an individual atomic transition, the states with $n=2,3,\ldots N$ or noncollective solutions of the TDA equations. If one looks at the complete Hamiltonian, in addition to the TDA sector, and now replaces the expression of S_+ and S_- in Eq. (1) one gets

$$H_{TDA} = -\omega_f \Omega N + W_1 \gamma_1^\dagger \gamma_1 + \sum_{n=2}^{N} W_n \gamma_n^\dagger \gamma_n + \omega_b (a^\dagger a + \tfrac{1}{2}) + \zeta \sqrt{2\Omega N}(a^\dagger \gamma_1 + \gamma_1^\dagger a). \tag{17}$$

This version of the Hamiltonian contains the boson contributions of the external photon field, the effective boson $\gamma_1^\dagger$, a vacuum energy, the contribution of the noncollective bosons, $\gamma_n^\dagger$, and the interaction of the external photon and the coherent superposition $\gamma_1^\dagger$. As a first approximation we shall neglect the contributions of $\gamma_n^\dagger$, and look for the solution of the remaining Hamiltonian. The states

$$|l,k\rangle = N_{l,k} a^{\dagger l} \gamma_1^{\dagger k} |0\rangle, \tag{18}$$

are eigenstates of the symmetry operator

$$O_I = a^\dagger a + \gamma_1^\dagger \gamma_1 + \tfrac{1}{2} N, \tag{19}$$

since

$$O_I |l,k\rangle = L |l,k\rangle^{(L)}, \quad L = l + k. \tag{20}$$

The matrix elements of the Hamiltonian of Eq. (17) on the basis of Eq. (18) are given by

$$\begin{aligned} \langle lk|H_{TDA}|lk\rangle^{(L)} &= \omega_f (k - \tfrac{1}{2}N) + \omega_b (L - k + \tfrac{1}{2}) + 2\lambda k(N-1), \\ \langle l+1\,k-1|H_{TDA}|lk\rangle^{(L)} &= \zeta \sqrt{N}\sqrt{(L-k+1)k}, \\ \langle l-1\,k+1|H_{TDA}|lk\rangle^{(L)} &= \zeta \sqrt{N}\sqrt{(L-k)(k+1)}, \end{aligned} \tag{21}$$

where, as in Eq. (9), we have used the value $2\Omega = 1$.

Note that the nondiagonal matrix elements of H_{TDA}, Eq. (21), exhibit a distinctive dependence on $\zeta\sqrt{N}$, which in this case is a direct consequence of the TDA approach. The appearance of the factor $\zeta\sqrt{N}$ has been noticed previously in Refs. 2,4. In fact, in Refs. 2,4 it was found that the group of N junctions behaves somewhat like a single junction with the coupling to the cavity mode $\zeta\sqrt{N}$ instead of ζ.

D. Approximate solution: The Boson mapping

In the preceding section we have expressed the dominant part of the Hamiltonian in a certain restricted subspace, which is the subspace of particle-hole-like excitation $\gamma_n^\dagger$. Another option, in order to obtain approximate solutions of the Hamiltonian of Eq. (1), is to express the fermionic operators in terms of bosonic ones. We also demand that the algebraic structure of operators entering in the Hamiltonian remains invariant after performing the mapping to the bosonic representation. Since the algebra obeyed by $S_\pm$ and S_z is a su(2) algebra, we shall look at the expression of these operators in terms of boson operators $b^\dagger$ and b, such that (i) $[b,b^\dagger]=1$, and (ii) the transformed operators $S_\pm(b,b^\dagger)$ and $S_z(b,b^\dagger)$ obey the commutation relations of Eq. (3). There are several possible boson mappings.[9] Among them we have adopted the Holstein-Primakoff boson mapping:

$$\begin{aligned} S_+ &\rightarrow b^\dagger \sqrt{N - b^\dagger b}, \\ S_- &\rightarrow \sqrt{N - b^\dagger b}\, b, \\ S_z &\rightarrow b^\dagger b - \frac{N}{2}. \end{aligned} \tag{22}$$

Replacing the operators of Eq. (22) in the Hamiltonian of Eq. (1), and after expressing the interaction between fermions as a scalar product we obtain

$$H_{BM} \approx [\omega_f + \lambda(2N-3)] b^\dagger b + \omega_b (a^\dagger a + \tfrac{1}{2}) - \frac{N}{2}\omega_f - 2\lambda b^{\dagger 2} b^2 + \zeta (a^\dagger \sqrt{N - b^\dagger b}\, b + b^\dagger \sqrt{N - b^\dagger b}\, a). \tag{23}$$

TABLE I. Energy E_α, partial contributions to the total strength $B(S_z)$, and partial contributions to the EWSS for each eigenvalue index α. The exact solution (*exact*), is compared with the results of the TDA approximation, and with the results of the BM. N is the number of atoms, L is the eigenvalue of the symmetry operator, ζ is the strength of the coupling between the atoms and the photons, and λ is the strength of the atom-atom interaction. The values shown in this table have been obtained for $N=2$, $L=1$, $\zeta=-0.015$, and $\lambda=0.002$. The values E_α are given in units of $\hbar\omega_b$ (the energy of the cavity mode), the values $B(S_z)$ are given in units of $\hbar^2$ and the EWSS in units of $\hbar/\omega_b$. The quantities given at the end of the corresponding columns are the accumulated values of each of the sums $B(S_z)$ and EWSS.

α	$E_\alpha(exact)$	E_α(TDA)	E_α(BM)	$B(S_z)_{exact}$	$B(S_z)_{TDA}$	$B(S_z)_{BM}$	EWSS_{exact}	EWSS_{TDA}	EWSS_{BM}
1	0.48069	0.48069	0.47976	0.29914	0.29914	0.27410	0.00000	0.00000	0.00000
2	0.52331	0.52331	0.52224	0.24780	0.24780	0.24945	0.01056	0.01056	0.01059
				0.547	0.547	0.524	0.011	0.011	0.011

TABLE II. Same as Table I for $N=8$, $L=6$, $\zeta=-0.015$, and $\lambda=0.0$.

α	$E_\alpha(exact)$	E_α(TDA)	E_α(BM)	$B(S_z)_{exact}$	$B(S_z)_{TDA}$	$B(S_z)_{BM}$	EWSS_{exact}	EWSS_{TDA}	EWSS_{BM}
1	2.28486	2.24544	2.28486	2.61689	1.00000	2.61689	0.00000	0.00000	0.00000
2	2.36325	2.33029	2.36325	1.33342	1.50000	1.33342	0.10453	0.12728	0.10453
3	2.43431	2.41515	2.43431	0.01992	0.00000	0.01992	0.00298	0.00000	0.00298
4	2.50000	2.50000	2.50000	0.00030	0.00000	0.00030	0.00006	0.00000	0.00006
5	2.56569	2.58485	2.56569	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
6	2.63675	2.66971	2.63675	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
7	2.71514	2.75456	2.71514	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
				3.971	2.500	3.971	0.108	0.127	0.108

In the above expression some of the terms which are proportional to λ originate in the rearrangement of the fermionic interactions of the Hamiltonian of Eq. (1), as said before. By rearrangement we meant that the products of fermionic operators have been normal ordered (Wick theorem[8]) before applying the boson mapping (BM). We stress that, under the BM, the original spin-spin interaction in H can be interpreted in H_{BM} as a Kerr nonlinearity $b^{\dagger 2}b^2$ of the effective boson field. We recall that some features of Kerr nonlinearity in su(2) systems have been studied in Ref. 23, and a nonlinear Kerr Hamiltonian has been related with a mesoscopic Josephson junction model in Ref. 24. Also note that the initial photon-junction interaction in H is now substituted by an effective boson-boson term with intensity dependent coupling of the type $\sqrt{N-b^\dagger b}$.

Alternatively, one may directly apply the boson mapping to the complete Hamiltonian of Eq. (1) and work out a classification of terms in powers of the density $b^\dagger b$. Note that the TDA approximation presented in Sec. II C represents the leading order of such an expansion.

The purely bosonic Hamiltonian of Eq. (23) can be diagonalized in the basis

$$|l,k\rangle = \frac{1}{\sqrt{l!k!}} a^{\dagger l} b^{\dagger k}|0\rangle. \tag{24}$$

The matrix elements of the Hamiltonian of Eq. (23) are written as

$$\langle l,k|H_{BM}|l,k\rangle^{(L)} = \omega_f(k-\tfrac{1}{2}N)+\omega_b(L-k+\tfrac{1}{2}) + 2\lambda k(N-k-\tfrac{1}{2}),$$

$$\langle l+1,k-1|H_{BM}|l,k\rangle^{(L)} = \zeta\sqrt{(L-k+1)k(N-k+1)},$$

$$\langle l-1,k+1|H_{BM}|l,k\rangle^{(L)} = \zeta\sqrt{(L-k)(k+1)(N-k)} \tag{25}$$

with $L=l+k$.

E. Energy and non-energy-weighted strengths

If $\{E_\alpha\}$ denotes the complete set of eigenvalues of any of the Hamiltonians of the preceding sections, the energy-weighted strength sum (EWSS) is defined as

$$\mathrm{EWSS} = \sum_\alpha (E_\alpha - E_0)|\langle\alpha|S_z|0\rangle|^2. \tag{26}$$

The non-energy-weighted strength sum, also known as the strength function, is given by

$$B(S_z) = \sum_\alpha |\langle\alpha|S_z|0\rangle|^2, \tag{27}$$

and the time evolution of the population (e.g., number of atoms in the excited atomic state) is expressed by

$$S(t) = \mathrm{Tr}[\rho(t)(S_z+1/2)], \tag{28}$$

where $\rho(t)$ is the density matrix.

In order to gather information about the quality of the solutions obtained with the Hamiltonians H_{TDA} and H_{BM}, as compared with the exact solution of H in Eq. (1), we have computed the quantities EWSS, $B(S_z)$, and $S(t)$, by using the exact and approximate wave functions and energies.

TABLE III. Same as Table I for $N=8$, $L=6$, $\zeta=-0.015$, and $\lambda=0.002$.

α	$E_\alpha(exact)$	E_α(TDA)	E_α(BM)	$B(S_z)_{exact}$	$B(S_z)_{TDA}$	$B(S_z)_{BM}$	EWSS_{exact}	EWSS_{TDA}	EWSS_{BM}
1	2.32977	2.31594	2.32587	4.33696	3.76395	4.04660	0.00000	0.00000	0.00000
2	2.40680	2.40529	2.40126	1.32058	1.35271	1.35183	0.10171	0.12087	0.10192
3	2.47560	2.49465	2.46876	0.02671	0.00000	0.02908	0.00390	0.00000	0.00416
4	2.54018	2.58400	2.53290	0.00062	0.00000	0.00069	0.00013	0.00000	0.00014
5	2.60760	2.67335	2.60072	0.00001	0.00000	0.00001	0.00000	0.00000	0.00000
6	2.68243	2.76271	2.67627	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
7	2.76563	2.85206	2.76022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
				5.685	5.117	5.428	0.106	0.121	0.106

TABLE IV. Same as Table I for $N=80$, $L=6$, $\zeta=-0.015$, and $\lambda=0.0$.

α	$E_\alpha(exact)$	E_α(TDA)	E_α(BM)	$B(S_z)_{exact}$	$B(S_z)_{TDA}$	$B(S_z)_{BM}$	EWSS_{exact}	EWSS_{TDA}	EWSS_{BM}
1	−34.29239	−34.30499	−34.29239	1372.58057	1368.99976	1372.58057	0.00000	0.00000	0.00000
2	−34.02812	−34.03665	−34.02812	1.49890	1.49999	1.49890	0.39612	0.40250	0.39612
3	−33.76402	−33.76833	−33.76402	0.00016	0.00000	0.00016	0.00008	0.00000	0.00008
4	−33.50000	−33.50000	−33.50000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
5	−33.23598	−33.23168	−33.23598	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
6	−32.97188	−32.96334	−32.97188	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
7	−32.70761	−32.69501	−32.70761	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
				1374.080	1370.500	1374.080	0.396	0.402	0.396

III. RESULTS AND DISCUSSION

In this section we are going to present and discuss the results of our calculations. We have taken the number of atoms and the coupling strength (ζ and λ) as the parameters of the model. We have diagonalized the Hamiltonian H of Eq. (1) and solved the TDA and bosonic images of it (H_{TDA} and H_{BM}). The results are shown in Tables I–V. Some of the features which can be extracted from the analysis of the results are the following.

(a) Concerning the eigenvalues, the bosonic approximation H_{BM} compares very satisfactory (up to two digits) with the exact solution. The agreement between the linear approximation H_{TDA} and the exact solution is acceptable at the level of the first digit and it improves for larger values of N.

(b) The fragmentation of the strength, namely, the amount of each atomic excitation represented by each solution, and the partial contributions to the sum $B(S_z)$ and to EWSS, are better reproduced in the case of the BM approximation.

(c) The accumulated values of $B(S_z)$ and EWSS are better reproduced, as compared with the exact results, by the bosonic approximation.

(d) The linear TDA approach underestimates the total strength and it overestimates the energy-weighted sum rule.

(e) The agreement improves if N increases, particularly, when the interaction between sites is turned on.

In spite of these differences, the overall agreement of the spectra obtained by the TDA and BM approximations, with respect to the exact solution, is rather acceptable. However, in order to qualify the approximations, the information about the eigenvalues must be complemented by the corresponding values of the intensity and EWSS.

These results support the notion that both the algebraic treatment (boson expansion) and the linearization procedure (TDA) constitutes a valid alternative to the exact diagonalization of the N junctions interacting with an external optical phonon and among themselves. It also shows the suitability of the collective approach, which consists of replacing fermion-pair excitations by bosonic modes of excitation. In this respect, and considering also the appearance of the factor $\zeta\sqrt{N}$, the present formalism constitutes a natural extension of the treatment presented in Ref. 2. Moreover, as a by-product, the boson mapping technique can be used in order to relate the spin-spin interaction term between the junctions to a kind of nonlinear Kerr effect whose dynamical consequences could be worth studying, in particular those concerning quantum superposition phenomena.[23–25]

Another piece of information, which can be used to compare the different approximations, refers to the behavior of the response function $S(t)$. Figure 1 shows the results of $S(t)$ corresponding to the exact diagonalization (solid lines) and the ones of the boson expansion (dotted lines), for $N=2,4$ junctions. The initial state consists of a single-excited junction. Thus, $S(t)$ measures the time evolution of the occupation of a single junction. The agreement between the exact solution and the boson approximation is rather satisfactory. Figure 2 shows the results corresponding to a different initial state, which is the one consisting of a boson and all junctions in the lower-energy state. The time evolution of $S(t)$ from Fig. 2 shows a phase factor $\pi/2$, as compared with the time evolution of $S(t)$ from Fig. 1. Figures 3 and 4 shows the comparison between the results of the exact [inset (a)] TDA [inset (b)], and boson expansion [inset (c)] methods for

TABLE V. Same as Table I for $N=80$, $L=6$, $\zeta=-0.015$, and $\lambda=0.002$.

α	$E_\alpha(exact)$	$E_\alpha(TDA)$	$E_\alpha(BM)$	$B(S_z)_{exact}$	$B(S_z)_{TDA}$	$B(S_z)_{BM}$	$EWSS_{exact}$	$EWSS_{TDA}$	$EWSS_{BM}$
1	−33.79542	−33.79567	−33.79686	1543.24353	1543.45239	1542.75439	0.00000	0.00000	0.00000
2	−33.38527	−33.38111	−33.38829	0.63366	0.62844	0.63848	0.25989	0.26052	0.26087
3	−32.98071	−32.96656	−32.98528	0.00007	0.00000	0.00007	0.00006	0.00000	0.00006
4	−32.58156	−32.55200	−32.58764	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
5	−32.18764	−32.13744	−32.19518	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
6	−31.79874	−31.72289	−31.80772	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
7	−31.41466	−31.30833	−31.42503	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
				1543.877	1544.081	1543.393	0.260	0.261	0.261

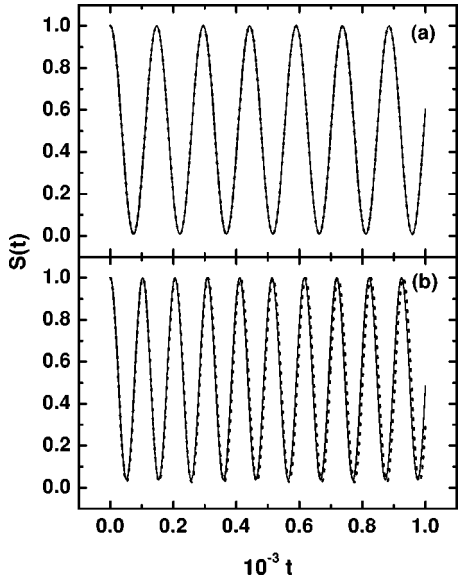

FIG. 1. Time evolution of the population of atoms in the excited state $S(t)$ as a function of t. The time is measured in units of ω_b^{-1}, and the results correspond to the case $N=2$ [inset (a)], and $N=4$ [inset (b)]. The initial state consists of a single atom in its excited state and all other junctions and the cavity boson in their ground states. Solid lines show the results of the exact diagonalization. Dotted lines show the results of the boson mapping of Sec. II D.

$S(t)$, with and without the inclusion of the site-site interaction (cf. Ref. 2, the extended Dicke model). As seen from these results, the agreement between the TDA approximation and the exact solution improves if the site-site interaction is present, but, in general, the agreement between the bosonic approximation and the exact solution is much better. Figures 5 and 6 show the time evolution of $S(t)$, for $N=8$ and $N=80$ junctions. In both cases, we have calculated $S(t)$ by considering $L=6$ bosons in the initial state and all junctions

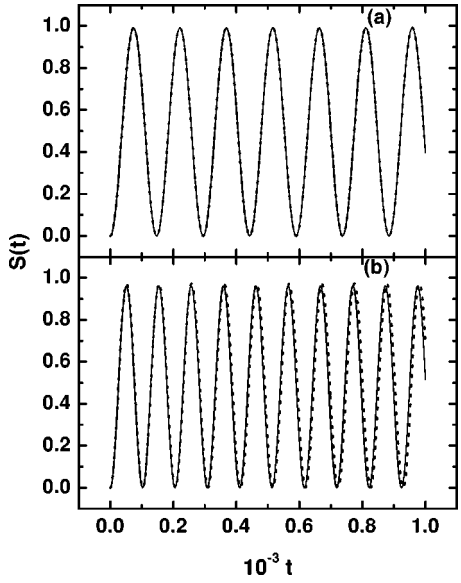

FIG. 2. Same as Fig. 1, with one cavity boson in the first excited state and all junctions in their ground states.

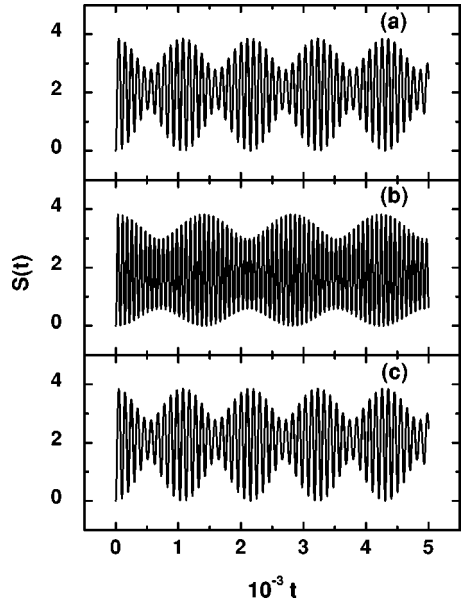

FIG. 3. $S(t)$ for $N=4$ junctions and $L=6$. The coupling between junctions is fixed at $\lambda=0$. The initial state has six photons in the resonator and all the junctions are in their ground states. Insets (a), (b), and (c) show the results of the exact treatment, the TDA, and the boson mapping, respectively.

in the lower-energy state. Only the exact results [insets (a) and (c)] and the ones of the boson expansion [insets (b) and (d)] are shown. The agreement between both sets of results is rather good over the extended time evolution.

IV. CONCLUSIONS

In this work we have studied some alternatives to the exact diagonalization of the problem posed by a system of N-Josephson junctions interacting with a photon field and among themselves. The conventional treatment makes use of

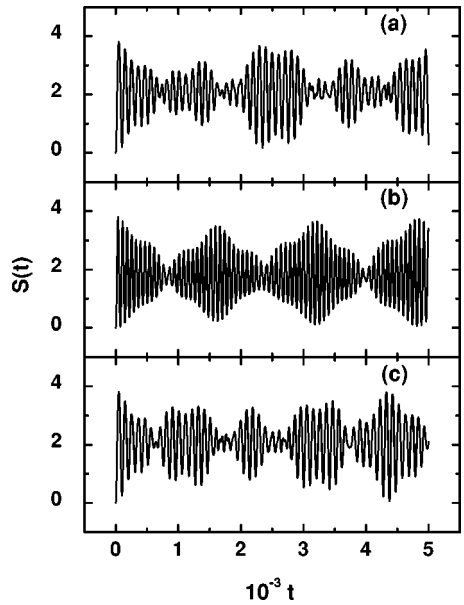

FIG. 4. Same as Fig. 3 for $\lambda=0.002$.

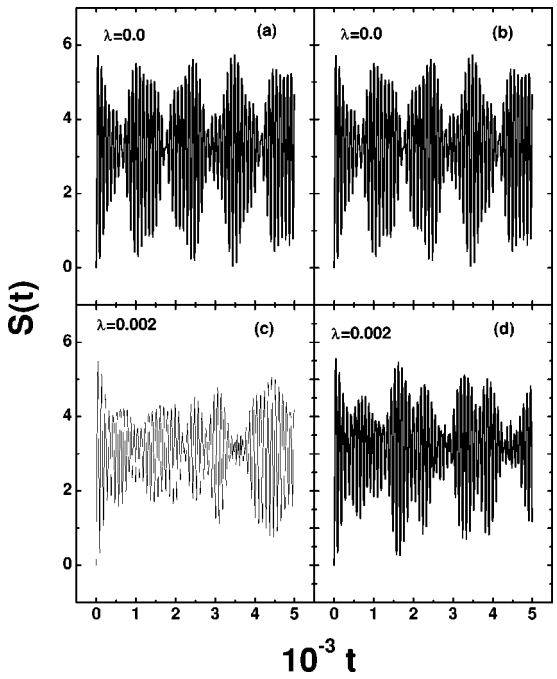

FIG. 5. $S(t)$ for the case $N=8$, $L=6$. The initial state has six photons in the first excited state and all the junctions in their ground state. Insets (a) and (c) show the exact results, and insets (b) and (d) show the results of the boson mapping. The values of the strength λ used in the calculations are indicated in each inset.

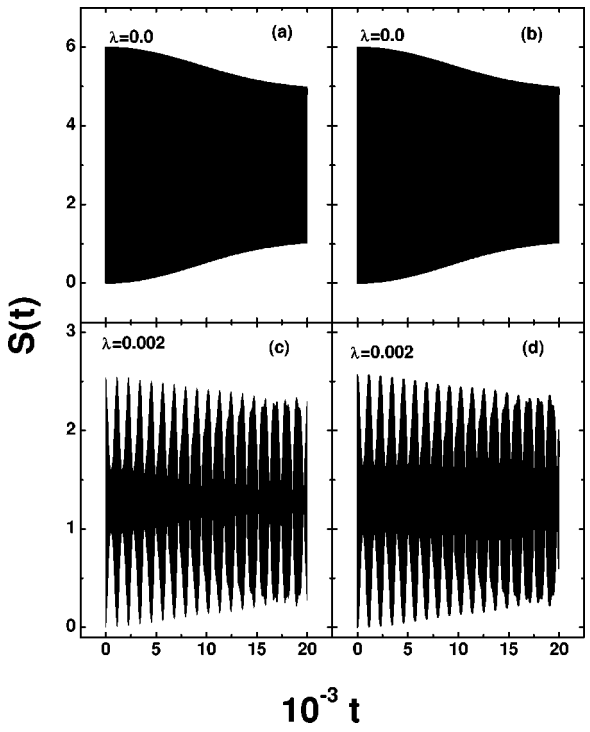

FIG. 6. Same as Fig. 5 for $N=80$ junctions.

the isomorphism between the atomic excitations, by the absorption and emission of the optical photon, and a quantum mechanical systems of N two-level, pseudospin-1/2-like qubits.[26,27] The Hamiltonian of the system belongs to the familiar Dicke's structure and its extended version, which includes a long-range site-site interaction. We have advanced the notion of a superposition between the photon and the excitation of the atomic levels, as in the case of a spin wave in the Heisenberg spin-spin interaction in presence of an external magnetic field. We have applied two different boson techniques, namely, (a) the TDA linear approximation, and (b) a boson expansion method. We have found that the boson expansion reproduces rather satisfactory, the exact, results, even for a relatively large number of junctions. The agreement between the TDA and exact results is not as good as that of the agreement between the exact and boson expansion methods, but still the TDA shows the correct trend of the exact solution. We have noted that the comparison of the eigenvalues does not suffice for a qualification of the approximations, and that the information about the strength function and EWSS provides a good indication about the validity of the approximations. This is also true for the case of the results corresponding to the time evolution of the population of atoms in the excited state. The time evolution of $S(t)$, over a large number of periods, shows that the boson expansion accurately describes the features of the exact solution.

In view of these results we conclude that the use of the boson expansion, in the treatment of Hamiltonians belonging to the class of Dicke's Hamiltonians may be a suitable alternative (and perhaps the only one applicable) to an exact diagonalization. From a more physical oriented point of view, the validity of the boson expansion underlines the collective structure of the motion induced by the scattering of the optical photon by the system of junctions.

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