Illustrations of the Becchi–Rouet–Stora–Tyutin invariance by means of simple toy models

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Among the available quantization methods for gauge field theories, the Becchi–Rouet–Stora–Tyutin (BRST) procedure has emerged as the simplest one for non-Abelian field theories, and thus the one that is systematically used in such cases. Our aim is to provide the reader with an accessible introduction to this modern and elegant treatment of constrained systems through its application to two simple mechanical models: a particle moving on a ring, and a particle on the surface of a sphere. If the description of these models is made from a moving frame of reference, they constitute simple analogs of gauge field theories. Notwithstanding their simplicity, these two applications display the main features of the BRST method, dealing with Abelian and non-Abelian symmetries, respectively. They also illustrate the solution of many-body problems in which broken symmetries are restored by means of collective coordinates describing the motion of the moving frame. Both the Hamiltonian and the Lagrangian formulations are presented, the latter using the antifield formalism.

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I. INTRODUCTION

In the everyday world, we may perform certain operations on physical objects which leave them unchanged. We call these operations *symmetry operations*. By extension, we may consider the mathematical operations acting on the entities of a given theory that leave invariant the corresponding physical laws.

There are *global* invariances, in which the same transformations are carried out at all space–time points, and the (more interesting) *local* invariances, in which different transformations are carried out at different points. An example of local transformations is in Maxwell's theory of electromagnetism. The electric and magnetic fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, may be obtained from the vector potential $\mathbf{A}(\mathbf{r},t)$ and the electrostatic potential $V(\mathbf{r},t)$. However, these potentials are not unique. The local transformations that $\mathbf{A}$ and V may undergo while preserving $\mathbf{E}$ and $\mathbf{B}$ are called gauge transformations and the associated invariance of Maxwell's equations is called gauge invariance.

We may turn the formulation around and derive electromagnetism from the construction of a gauge invariant Lagrangian under the local transformations of the particle field

$$\Psi(\mathbf{r},t)\rightarrow e^{ie\Lambda(\mathbf{r},t)}\Psi(\mathbf{r},t), \tag{1}$$

where e is the charge of the particle. The electromagnetic potentials, with adequate transformation properties, are introduced to satisfy the invariance requirement. Electromagnetism has been reborn as a manifestation of a local invariance.[1]

The transformations in Eq. (1) are called *Abelian* because two successive transformations may commute their sequence without altering the final result. In 1954 Yang and Mills used a non-Abelian group of transformations, the group of isotopic spin rotations.[2] The vector fields generalizing the photon field were interpreted as the fields of strongly interacting vector mesons of isotopic spin one. Gauge theories were further generalized to arbitrary non-Abelian gauge groups. It eventually turned out that all the observed interactions between elementary particles are generated by vector fields associated with local gauge symmetries. Typical examples are the standard models for electroweak and strong interactions.[1]

The fundamental tool presently used in the quantization of gauge fields is based on the Becchi–Rouet–Stora–Tyutin (BRST) invariance,[3,4] because of its simplicity in non-Abelian cases. However, this simplicity is only relative to other procedures. Our aim is to contribute to the understanding of this procedure through its application to simple systems. The possibility of such applications stems from the fact that "a gauge theory may be thought of as one in which the dynamical variables are specified with respect to a reference frame whose choice is arbitrary at every instant of time."[4] This characterization applies, in particular, to mechanical systems that are described from a moving frame of reference. Mechanical systems may be considered as field systems in $(n+1)$ dimensions, where n, the number of spatial dimensions, equals 0. Moreover, if their description is made from a moving frame of reference, they are the analogs of gauge field systems, as shown in Sec. II. The analogy between gauge fields and mechanical systems described from moving frames of reference is extensively exploited in Ref. 5.

In contrast to the very general and complete treatment of Ref. 4 (in which few applications, if any, are made), we confine our discussion to two simple toy models. The first one consists of a particle of mass m that is allowed to move along a ring (Sec. II). This model already illustrates many important features of the BRST procedure, which is outlined in Sec. III and applied to the simple model in Sec. IV. The second model, the motion of a particle on the surface of a sphere (Sec. V), displays in addition the non-Abelian complications that are solved in such an elegant way through the BRST procedure. These two applications are made using the

Hamiltonian formulation. As mentioned at the end of Sec. IV, the BRST procedure is too powerful for the treatment of the Abelian case and, as a consequence, some essential degrees of freedom appear to be uncoupled (and thus unused) in the final BRST Hamiltonian. On the other hand, many features, which are also present in non-Abelian problems, may be understood already from the Abelian case. We illustrate the use of the Lagrangian formalism in Sec. VI.

The formalism applied in this paper is purely algebraic and thus, in particular, it does not rely on functional methods. These methods constitute the framework of the usual presentations. Their use allows to develop the full power of the BRST procedure.

II. THE ABELIAN TOY MODEL

Let us consider a particle of mass m that is allowed to move along a ring of radius r_0. The model thus has initial cylindrical symmetry. Although the solution of this problem is simple enough in the laboratory frame, it becomes even simpler in a system of reference rotating with the particle. Such a frame may be defined, for instance, by the condition

$$y=0. \tag{2}$$

In this rotating frame of reference, the solution of the problem (not merely the initial conditions) is written simply as

$$x=r_0, \quad p_x=0. \tag{3}$$

However, this solution violates the original cylindrical symmetry of the problem, and thus it may be characterized as being a *deformed* solution. Moreover, there is no restoring force in the y direction. The absence of angular motion in the description of the moving frame must be compensated by the introduction of the angular coordinate ϕ, describing the motion of the rotating frame relative to the laboratory frame. Thus, in addition to the x degree of freedom, there is an overcomplete set of variables associated with the angular motion (y,ϕ).

From here on we label as *collective* coordinates the ones relating the motion of the moving frame to the laboratory, and as *intrinsic* coordinates those describing the motion with respect to the moving frame.

Let us further examine the present scenario. The intrinsic coordinates x,y are related to the coordinates in the laboratory system $x_{\rm lab},y_{\rm lab}$ by (see Fig. 1)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_{\rm lab} \\ y_{\rm lab} \end{pmatrix}. \tag{4}$$

We take D_t to be the total time derivative of the position of the particle, and a dot to be the time derivative of the position with respect to the moving frame. Using Eq. (4), we obtain

$$D_t x = \dot{x} + \frac{\partial x}{\partial\phi}\dot{\phi} = \dot{x} + \dot{\phi} y, \tag{5a}$$

$$D_t y = \dot{y} + \frac{\partial y}{\partial\phi}\dot{\phi} = \dot{y} - \dot{\phi} x. \tag{5b}$$

Here D_t and $\dot{\phi}$ are the analogs of a covariant derivative and of a gauge field, respectively. The Lagrangian is[6]

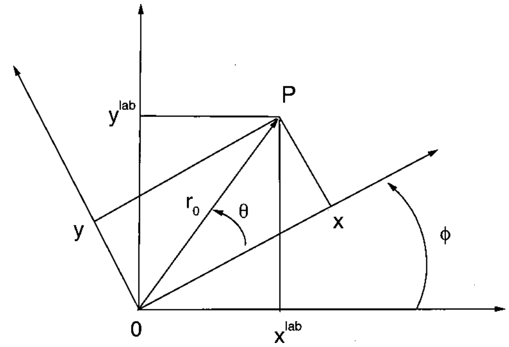

Fig. 1. Intrinsic (x,y) and laboratory $(x_{\rm lab},y_{\rm lab})$ coordinates of the generic point P on a ring of radius r_0. The two set of coordinates are related by a transformation [see Eq. (4)], which depends only on the angular coordinate ϕ. Readers can easily convince themselves of the fact that the transformation between the intrinsic and laboratory frame does not depend explicitly on the value of the angle θ.

$$L = \frac{m}{2}(D_t x)^2 + \frac{m}{2}(D_t y)^2 = \frac{m}{2}(\dot{x}+\dot{\phi} y)^2 + \frac{m}{2}(\dot{y}-\dot{\phi} x)^2. \tag{6}$$

Such a Lagrangian is called *singular* because the determinant of the matrix $\partial^2 L/\partial z_i \partial z_j$ $(z_i = \dot{x},\dot{y},\dot{\phi})$ vanishes. There are three equations for the momenta, namely

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m(\dot{x}+\dot{\phi} y), \tag{7a}$$

$$p_y \equiv \frac{\partial L}{\partial \dot{y}} = m(\dot{y}-\dot{\phi} x), \tag{7b}$$

$$I \equiv \frac{\partial L}{\partial \dot{\phi}} = x p_y - y p_x \equiv l. \tag{7c}$$

Here I is the generator of collective rotations, while l can be recognized as the generator of intrinsic rotations. Although Eqs. (7a) and (7b) yield the momenta p_x,p_y as a function of the velocities, Eq. (7c) is only a relation between the three momenta. Thus we are not able to solve simultaneously the three equations of motion. This failure can be traced to the fact that the Lagrangian describing the system from the moving frame does not contain information about the motion of the frame itself. Instead, we obtain two momenta and a constraint

$$F \equiv l - I = 0. \tag{8}$$

Equation (8) expresses the obvious fact that if the particle is rotated by any angle relative to the moving frame, the corresponding description should be completely equivalent to the one obtained by rotating the moving frame by the opposite angle.

The existence of constraints is also central to the analogy between gauge field theories and mechanical models that are described from moving frames. Within the extended phase space defined by the intrinsic and collective variables, and their conjugate momenta, physical trajectories are restricted to a hypersurface defined by the constraint (8). On this hypersurface there are families of trajectories that transform into each other using F as a generator. Indeed, F represents the generator of the gauge transformations that leave the system invariant. To select a gauge is to select one trajectory from the family of equivalent trajectories.

The physical variables are those that are independent of the local reference frame: physical variables (''observables'') are said to be gauge-independent.

Our aim is to quantize this classical model. The following commutation relations hold:

$$[x,p_x]=[y,p_y]=[\phi,I]=i, \tag{9}$$

where $\hbar=1$ from here on. Because we have introduced new variables that enlarge the vector space, we must expect the presence of *unphysical* states and operators, in addition to *physical* ones. The constraint (8) is equivalent to the quantal conditions

$$F|\mathrm{ph}\rangle=0, \quad F|\mathrm{unph}\rangle\neq 0, \tag{10a}$$

$$[F,O_{\mathrm{ph}}]=0, \quad [F,O_{\mathrm{unph}}]\neq 0, \tag{10b}$$

where the label ph (unph) indicates physical (unphysical) states or operators. However, the separation between physical and unphysical states is by no means a trivial operation, except in simple examples. In Sec. III we present a systematic procedure to accomplish the separation.

III. OUTLINE OF THE BRST SOLUTION FOR ABELIAN CASES

In the previous example, the existence of an overcomplete set of two variables is compensated by the presence of a constraint. The most natural thing to do would be to use the constraint to reduce the number of variables to the initial number. However, ''it is a remarkable occurrence that the road to progress has invariably been toward enlarging the number of variables and introducing a more powerful symmetry, rather than conversely aiming at reducing the number of variables and eliminating the symmetry.'' [4] In this section we describe how this enlargement may be performed.

According to Dirac,[7] we may replace the original Hamiltonian by

$$H_{\mathrm{Dirac}}\equiv H-\lambda F, \tag{11}$$

which yields the same results as H for physical states. Dirac introduced an additional boson variable, the Lagrange multiplier λ, with B as its conjugate partner ($[\lambda,B]=i$). The new constraint,

$$B=0, \tag{12}$$

should hold in order to maintain the problem associated with the original Hamiltonian. We do not attempt to solve the problem à la Dirac here.

Becchi, Rouet, Stora, and Tyutin[3] went further in this direction by adding two fermion variables $\eta,\overline{\eta}$ and their conjugate operators $\pi,\overline{\pi}$. These Hermitian, Grassman operators are called *ghosts*. Thus, they satisfy the anticommutation relations

$$\{\eta,\pi\}=\{\overline{\eta},\overline{\pi}\}=1, \tag{13a}$$

$$\{\eta,\overline{\eta}\}=\{\pi,\overline{\pi}\}=\{\eta,\overline{\pi}\}=\{\overline{\eta},\pi\}=0, \tag{13b}$$

$$\{\eta,\eta\}=\{\overline{\eta},\overline{\eta}\}=\{\pi,\pi\}=\{\overline{\pi},\overline{\pi}\}=0. \tag{13c}$$

The complication introduced by the addition of new variables is only apparent, because there appears a new supersymmetry involving boson and ghost variables. It completely captures the original gauge invariance of the problem [Eq. (8)] and leads to a simpler formulation of the theory.

The name ghost stems from the fact that they only appear in closed loops, that is, they can neither be experimentally produced or detected as free particles.[1] Although the concept of ghosts was previously applied in quantum physics, it is only with the advent of the BRST symmetry that they were raised to the same prominence as other variables or fields, being mixed with them.

A fundamental quantity in the BRST formalism is the charge Q, constructed as an operator linear in the constraints

$$Q\equiv -\eta F+\overline{\pi}B. \tag{14}$$

In the same way that F is the generator of gauge transformations, Q may be considered as the generator of BRST transformations. Because the constraints (8) and (12) imply that

$$Q=0, \tag{15}$$

we should stay within a subspace invariant under BRST transformations.

The operator Q is nilpotent ($Q^2=0$) and Hermitian ($Q^\dagger=Q$). Similar to the situation described at the end of Sec. II, there are now physical and unphysical states and operators. Indeed, Eqs. (10a) and (10b) continue to be valid upon the replacement of the constraint F by the charge Q. However, within the BRST formalism, there appear a new set of states

$$|\chi\rangle\equiv Q|\mathrm{unph}\rangle. \tag{16}$$

Using the properties of Q, it is straightforward to demonstrate that

$$Q|\chi\rangle=0, \quad \langle\chi|\chi\rangle=0, \tag{17}$$

that is, the $|\chi\rangle$ states are also annihilated by Q and have zero norm. One may also define operators O_χ, mapping physical states onto zero-norm states,

$$O_\chi\equiv\{Q,O_{\mathrm{unph}}\}, \quad O_\chi|\mathrm{ph}\rangle=|\chi\rangle. \tag{18}$$

As a consequence, it is possible to add $|\chi\rangle$ states to physical states and O_χ operators to physical operators,

$$|\mathrm{ph}\rangle\rightarrow|\mathrm{ph}\rangle+|\chi\rangle, \quad O_{\mathrm{ph}}\rightarrow O_{\mathrm{ph}}+O_\chi. \tag{19}$$

Again making use of the properties of Q, we easily verify that

$$\begin{aligned}(\langle(\mathrm{ph})_f|+\langle\chi_f|)(O_{\mathrm{ph}}+O_\chi)(|(\mathrm{ph})_i\rangle+|\chi_i\rangle)\\ =\langle(\mathrm{ph})_f|O_{\mathrm{ph}}|(\mathrm{ph})_i\rangle,\end{aligned} \tag{20}$$

that is, the transformations (19) do not change the value of the physical matrix elements. We construct the BRST Hamiltonian as a particular application of this freedom, by adding to H an O_χ operator

$$H_{\mathrm{BRST}}\equiv H+\{\rho,Q\}. \tag{21}$$

For any choice of the operator ρ, H_{BRST} yields the same physical eigenvalues as the original H.

The enlarged phase space, discussed at the end of Sec. II, has been further extended by the inclusion of Lagrange multipliers and ghost variables and their respective momenta. Physical trajectories are now restricted by the condition (15). Within the physical trajectories, there are families of equivalent trajectories that are obtained from each other using Q as the generator of the BRST transformations or, equivalently, by varying ρ. Thus, the selection of ρ is equivalent to the adoption of a gauge. One possible choice is motivated by an analogy with the covariant gauge in Yang–Mills theory,

$$\rho = \lambda\pi + \bar{\eta}\left(\theta - \frac{1}{2k}B\right). \tag{22}$$

In this case, Eq. (21) yields[8]

$$H_{\mathrm{BRST}} = H - \lambda F + i\pi\bar{\pi} + B\theta - \frac{1}{2k}B^2 + \eta\bar{\eta}[\theta, F], \tag{23}$$

where θ, a function of the intrinsic variables, should not commute with the constraint F, and thus with the intrinsic generator l. The gauge has not been completely fixed yet because, aside from the last requirement, we are still free to choose either the constant k or the function θ [see Eqs. (25) and (28) below]. The uncoupling of the different modes in Eq. (23), as well as the treatment of the negative "kinetic energy" term $-B^2/2k$, is also deferred to Sec. IV.

Because the Hamiltonian H_{BRST} (unlike H) does not commute with the intrinsic generator, the microscopic cylindrical symmetry is lost. On the other hand, H_{BRST} commutes with the collective generator: microscopic invariance has been replaced by macroscopic collective invariance.

IV. APPLICATION TO THE ABELIAN TOY MODEL

So far we have outlined a possible version of the BRST formalism that is suitable for general Abelian transformations. Let us now confine ourselves to its application to the toy model. The collective space associated with it is given naturally by the eigenfunctions of the angular momentum in two dimensions, namely

$$\Psi_{L_0}(\phi) = \frac{1}{\sqrt{2\pi}} e^{iL_0\phi}. \tag{24}$$

The collective degree of freedom ϕ, which was introduced in Sec. II as an artifact associated with the existence of a moving frame, has been raised to the level of a real degree of freedom.

Because in the present problem there is only one real degree of freedom, and this role is taken by the collective one [see Eq. (24)], the remaining ones are unphysical. Thus a trade off has taken place: the intrinsic degree of freedom y, associated with the angular motion, has been transferred to the unphysical subspace. In the following we discuss in some detail the structure of this subspace.

The classical, deformed solution (3) is taken to be the starting point for the motion in the intrinsic reference frame. The radius r_0 constitutes the large magnitude of the problem (the *order parameter*) and thus is much larger than any other length magnitude. As part of the process of selecting a gauge, we choose θ to be the conjugate variable to the leading order term of l $(=r_0 p_y)$,

$$\theta = y/r_0. \tag{25}$$

With this choice the moving frame becomes anchored to the particle, because the mean value $\langle y\rangle = 0$ [cf. the classical value in Eq. (2)],

$$H_{\mathrm{BRST}} = H_b + H_g + H_c + H_r, \tag{26a}$$

$$H_b \equiv \frac{1}{2m}p_y^2 - r_0\lambda p_y + \frac{1}{r_0}By - \frac{1}{2k}B^2, \tag{26b}$$

$$H_g \equiv i\pi\bar{\pi} + i\eta\bar{\eta}, \tag{26c}$$

$$H_c \equiv \lambda I, \tag{26d}$$

$$H' \equiv \frac{1}{2m}p_x^2 + \lambda l' + \frac{1}{r_0}x'\eta\bar{\eta}, \tag{26e}$$

where $x' \equiv x - r_0$ and $l' = x'p_y - yp_x$.

As is usual in field theory, we proceed first to diagonalize the quadratic Hamiltonian [Eqs. (26b) and (26c)], in order to define a basis in terms of independent bosons and fermions. The remaining Hamiltonian should be treated in perturbation theory.

By completing squares in H_b, we obtain

$$H_b = \frac{1}{2m}(p_y - mr_0\lambda)^2 + \frac{m}{2}y^2 - \frac{1}{2k}\left(B - \frac{k}{r_0}y\right)^2 - \frac{k}{2}\lambda^2. \tag{27}$$

We end the selection of the gauge by choosing

$$k = mr_0^2, \tag{28}$$

the moment of inertia of the system. With this choice $[(p_y - mr_0\lambda), (B - (k/r_0)y)] = 0$, the two kinetic energy contributions in H_b become mutually independent. The completion of the squares has yielded, in addition, restoring forces to the two oscillators in Eq. (27). Therefore, we may write H_b in terms of two uncoupled bosons, namely

$$H_b = \Gamma_1^\dagger\Gamma_1 - \gamma_0^\dagger\gamma_0, \tag{29}$$

where $\Gamma_1^\dagger$ and $\gamma_0^\dagger$ are defined as

$$\Gamma_1^\dagger \equiv \frac{1}{\sqrt{2m}}(p_y - mr_0\lambda) + i\sqrt{\frac{m}{2}}y, \tag{30a}$$

$$\gamma_0^\dagger \equiv -i\sqrt{\frac{1}{2k}}(B - mr_0 y) + \sqrt{\frac{k}{2}}\lambda, \tag{30b}$$

$$[\Gamma_1, \Gamma_1^\dagger] = [\gamma_0, \gamma_0^\dagger] = 1. \tag{30c}$$

The expression for H_b displays two oscillators: one has a positive frequency $(+1)$, the other a negative one (-1). This fact is unpleasant, not the least because the ground state becomes many times degenerate. To get out of this inconvenience, we may replace $\Gamma_0 \equiv \gamma_0^\dagger$, $\Gamma_0^\dagger = \gamma_0$, and therefore write

$$[\Gamma_0^\dagger, \Gamma_0] = 1, \tag{31a}$$

$$H_b = \Gamma_1^\dagger\Gamma_1 - \Gamma_0^+\Gamma_0 + 1. \tag{31b}$$

If the (new) vacuum state is annihilated by Γ_0 [cf. Eq. (36) below], all excitations of H_b become positive, at the expense of working with the metric (31a) for the 0-boson.[9] Note the difference from the usual metric (30c).

The ghost sector may be written as

$$H_g = \bar{a}a + \bar{b}b - 1, \tag{32}$$

where the following substitutions have been made:

$$a = i\bar{b}^\dagger \equiv \frac{1}{\sqrt{2}}(\bar{\pi} - i\eta), \tag{33a}$$

$$b = -i\bar{a}^\dagger \equiv \frac{1}{\sqrt{2}}(\pi + i\bar{\eta}), \tag{33b}$$

$$\{\overline{a},a\}=\{\overline{b},b\}=1. \tag{33c}$$

Therefore the quadratic Hamiltonian of the unphysical sector,

$$H_{\mathrm{unph}}\equiv H_b+H_g=\Gamma_1^\dagger\Gamma_1-\Gamma_0^\dagger\Gamma_0+\overline{a}a+\overline{b}b, \tag{34}$$

is a supersymmetric Hamiltonian with a characteristic excitation energy equal to unity. The eigenvectors of H_{unph} define the subspace

$$|n_1,n_0,n_a,n_b\rangle\equiv\frac{(\Gamma_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(\Gamma_0^\dagger)^{n_0}}{\sqrt{n_0!}}\overline{a}^{n_a}\overline{b}^{n_b}|\rangle, \tag{35}$$

where $n_1,n_0=0,1,2,\dots$ and $n_a,n_b=0,1$.

The vacuum state in this subspace satisfies the conditions

$$\Gamma_1|\rangle=\Gamma_0|\rangle=a|\rangle=b|\rangle=0. \tag{36}$$

We immediately verify that the quadratic charge in Eq. (14),

$$Q=-i(\Gamma_1^\dagger+\Gamma_0^\dagger)a-(\Gamma_1+\Gamma_0)\overline{b}, \tag{37}$$

annihilates the vacuum state and, because this state is normalizable, we conclude that it is a physical state. In fact, it is the only physical state of the subspace (35). The excited states (35) are paired: they are either unphysical states $|A\rangle$ or have zero norm $B\equiv Q|A\rangle$. For instance, the one-boson state $\Gamma_1^\dagger|\rangle$ is unphysical because $Q|n_1=0\rangle=-|n_b=1\rangle\neq 0$, while the state $|n_b=1\rangle$ has zero norm, because $\langle|\overline{b}^\dagger\overline{b}|\rangle=-i\langle|a\overline{b}|\rangle=0$ [cf. Eq. (33a)].

All the effects of the unphysical degrees of freedom on any physical magnitude must cancel out. This cancellation would not be possible to achieve if the unphysical subspace (35) were made up from ordinary states in the Hilbert space. The present cancellation is due to the unusual properties of the unphysical sector: the commutation relation (31a) of the boson labeled by 0, and the fact that neither of the fermion creation operators is the adjoint of the annihilation operator [$\overline{a}\neq a^\dagger$, $\overline{b}\neq b^\dagger$, cf. Eqs. (33a) and (33b)]. Nevertheless. all these operators are well defined and may be used without problems in subsequent mathematical manipulations.

As a consequence of the cylindrical symmetry, the original problem displays no restoring force in the y direction. In other words, the existence of a zero-frequency mode leads to the well-known problem of infrared divergencies, which prevents the straightforward application of perturbation theory. However, we mentioned at the end of Sec. III that the cylindrical symmetry is lost at the intrinsic level. As a consequence, all the unphysical degrees of freedom have a finite frequency [Eq. (34)], and thus perturbation theory has become feasible. This is a very important result of the formalism. The small parameter in the perturbation expansion can be chosen to be r_0^{-1}.

We suggest that the reader perform, as an exercise, some perturbative calculations. An interesting one is the following: there is a link between the collective and the intrinsic sectors (24) and (34), given by the term H_c in Eq. (26d), which is a Coriolis interaction. By inverting Eq. (30b), we obtain

$$\lambda=\frac{1}{\sqrt{2k}}(\Gamma_0^\dagger+\Gamma_0). \tag{38}$$

Because H_c is a small term $O(r_0^{-1})$, it may be treated perturbatively. Second-order perturbation theory yields

$$\Delta(L_0)=-L_0^2\langle|\lambda|n_0=1\rangle\langle n_0=1|\lambda|\rangle=\frac{1}{2k}L_0^2. \tag{39}$$

The correct moment of inertia has reappeared, yielding the correct rotational energy. In Eq. (39) we have used the value equal to unity for the intermediate excitation energy [cf. Eq. (34)]. The fact that second-order perturbation theory yields a positive contribution to the ground state is due to the unusual metric associated with the zero-phonon. Another instructive example is the calculation of the expectation value $\langle l^2\rangle=0$.

The residual interaction in H' vanishes in the limit of a rigid radial motion. However, we may allow some motion in the x direction, as would be the case for a Mexican hat potential. The Hamiltonian would display a physical, intrinsic degree of freedom with a finite frequency, in addition to the physical collective degree of freedom (24) and to the unphysical sector (35).

The BRST treatment of an Abelian model is somewhat of an overkill. Indeed, the ghost degrees of freedom appear to be completely decoupled in the Hamiltonian (26a) and, as a consequence, the ghosts never appear in the calculation, not even in intermediate states. Nevertheless, through the Abelian calculation we have been able to discuss properties of the BRST procedure that continue to be present in the non-Abelian case, such as the BRST symmetry, the existence of the zero-norm subspace, the construction of the unphysical subspace, and the feasibility of the perturbation expansion.

V. THE NON-ABELIAN TOY MODEL

In analogy to the Abelian case, the non-Abelian model consists of a particle that is allowed to move on the surface of a sphere of radius r_0. We define an intrinsic system by the conditions

$$x=y=0, \tag{40}$$

and thus the classical, deformed solution of the problem is

$$z=r_0,\quad p_z=0. \tag{41}$$

Although the spherical symmetry of the original problem has disappeared from Eqs. (40) and (41), the axial symmetry has been preserved.

The generators of the transformations on the surface of a sphere satisfy the commutation relations

$$[l_i,l_j]=i\epsilon_{ijk}l_k\quad \text{cyclical}\ (i,j,k), \tag{42}$$

where ϵ_{ijk} are the structure constants of the group of transformations. It is well known, since the early days of the quantum treatment of the rotations of molecules, that the corresponding collective generators satisfy the same commutation relations up to a sign, namely

$$[I_i,I_j]=-i\epsilon_{ijk}I_k. \tag{43}$$

Therefore, the three constraints satisfy the same commutators as the generators (42), namely

$$F_i\equiv l_i-I_i,\quad [F_i,F_j]=i\epsilon_{ijk}F_k, \tag{44}$$

and thus they commute within the physical subspace.

One of the most powerful attributes of the BRST procedure is to take into account the geometry of the gauge transformations. Accordingly, the BRST charge displays an additional term including the structure constants of the group of transformations. Note that there is no freedom in the determination of the charge, contrary to the arbitrariness in the selection of a gauge,

$$Q \equiv -\eta_i F_i + \overline{\pi}_i B_i + i\epsilon_{ijk}\eta_i\eta_j\pi_k. \tag{45}$$

The ghost operators carrying different subindices anticommute. The properties of Q stated in Sec. III remain valid in the non-Abelian case.

The collective sector is described by the three Euler angles α,β,γ. A complete set of states is given by the (rotational) eigenfunctions of the axial-symmetric top,

$$\Psi_{L_0MK} = \sqrt{\frac{2L_0+1}{8\pi^2}} D^{L_0}_{MK}(\alpha,\beta,\gamma), \tag{46}$$

where M and K are the projections of the collective angular momentum I over the laboratory and intrinsic z axis, respectively. Because our system consists of a particle, it cannot rotate around an axis passing through itself, and thus the eigenvalues of the operators l_z and I_z must be zero. Therefore, the rotational functions reduce to the spherical harmonics

$$\Psi_{L_0M(K=0)} = Y_{L_0M}(\beta,\alpha). \tag{47}$$

The disappearance of γ expresses the well-known fact that it is not possible to collectively treat the rotation around a symmetry axis. Within the present framework this statement implies that we should not fix the orientation of the moving frame with respect to rotations around this axis. Thus we generalize Eq. (21) only to the x,y directions

$$\rho = \pi_\nu \lambda_\nu + \overline{\eta}_\nu\left(\theta_\nu - \frac{1}{2k}B_\nu\right), \tag{48}$$

where $\nu = x,y$ and, therefore, $i = \nu,z$. We again make use of the expansion in $1/r_0$ in order to separate the leading order terms in l_i, namely

$$l_x = -r_0 p_y + l'_x, \quad l_y = r_0 p_x + l'_y, \quad l_z = l'_z. \tag{49}$$

We choose the θ_ν as the conjugate variable to the leading order terms in l_ν,

$$\theta_x = -\frac{1}{r_0}y, \quad \theta_y = \frac{1}{r_0}x. \tag{50}$$

The resulting Hamiltonian reads[8]

$$H_{\mathrm{BRST}} \equiv H + \{\rho, Q\} \tag{51a}$$

$$= \frac{1}{2m}p_\nu^2 - \lambda_\nu l_\nu + \theta_\nu B_\nu - \frac{1}{2k}B_\nu^2 + i\pi_\nu\overline{\pi}_\nu + i\eta_\nu\overline{\eta}_\nu + \frac{1}{2m}p_z^2 + \lambda_v I_v + \eta_i\overline{\eta}_\nu[\theta_\nu, l'_i] + i2\epsilon_{\nu ij}\lambda_\nu(\eta_i\pi_j - \eta_j\pi_i). \tag{51b}$$

The quadratic Hamiltonian is obtained by replacing the operators l_ν in line (51b) by their leading order expressions (49). At this level, the Hamiltonian is separable in the x,y degrees of freedom. For each ν, the terms are of the same form as $H_{\mathrm{unph}} = H_b + H_g$ of the Abelian case, cf. Eqs. (26b) and (26c). Therefore the same considerations may be applied here, leading to the supersymmetric quadratic Hamiltonian

$$H_{\mathrm{unph}} = \Gamma^\dagger_{\nu 1}\Gamma_{\nu 1} - \Gamma^\dagger_{\nu 0}\Gamma_{\nu 0} + \overline{a}_\nu a_\nu + \overline{b}_\nu b_\nu. \tag{52}$$

Quite generally, there will be as many contributions to the Hamiltonian of the quadratic type (34), as the number of collective variables that have been introduced.

The remaining contributions,

$$H'_{\mathrm{BRST}} = \lambda_\nu I_\nu - \lambda_\nu l'_\nu + \eta_i\overline{\eta}_\nu[\theta_\nu, l'_i] + i2\epsilon_{\nu ij}\lambda_\nu(\eta_i\pi_j - \eta_j\pi_i), \tag{53}$$

may be taken into account by means of perturbation theory. As discussed at the end of Sec. IV, this is only possible because the zero-frequency modes have been eliminated from the unperturbed spectrum (52).

Note that the ghost variables cannot be totally decoupled as in the Abelian case, even in the absence of radial motion. In particular, the ghost operators η_z, π_z are still coupled and no kinetic term is associated with them. Thus the ghost space may be split into two degenerate subspaces, corresponding to the two different states of the η_z ghost. However, there is no problem with the perturbation treatment, because the ghosts are always excited in pairs in which at least one of them has a finite energy [Eq. (52)].

The unperturbed vacuum state is also determined from conditions similar to Eq. (36). However, this vacuum state is only annihilated by the quadratic charge (37), but not by the higher order terms in Q. To the extent that the states are improved through the use of perturbation theory, the condition $Q|\mathrm{ph}\rangle = 0$ becomes more satisfied.

The perturbative treatment of the coupling term $\lambda_\nu I_\nu$ yields, once again, the rotational energy

$$-I_\nu^2\langle|\lambda_\nu|n_{\nu 0}=1\rangle\langle n_{\nu 0}=1|\lambda_\nu|\rangle = \frac{1}{2k}L_0(L_0+1). \tag{54}$$

VI. LAGRANGIAN FORMULATION. THE ANTIFIELD FORMALISM

The Hamiltonian BRST formalism leads to gauge-fixed actions that can be used within the path integral formalism. In principle, upon integration of the momenta, one may obtain a Lagrangian form. However, in many cases, it is difficult to obtain a manifestly relativistic path integral. Fortunately, there exists a systematic method for directly writing down the correct covariant gauge-fixed Lagrangian. It is known as the antifield approach to the BRST symmetry.[10,11] The antifield formalism is now considered to be the most powerful method for the quantization of gauge theories. It lets us treat problems other than those in which there is a Lie (closed) group of transformations, such as in Eq. (42). It may deal also with those situations in which the commutator between two generators yield, in addition to a third generator, terms that are proportional to the equations of motion (open algebras). Supergravity theories constitute an example.

For simplicity, we confine the presentation only to the Abelian model. We start from the classical Euclidean action (that is, we use the complex time $\tau = it$), obtained from the Lagrangian (6),

$$S' = \int d\tau\left(\frac{1}{2}(\dot{x} + \dot{\phi}y)^2 + \frac{1}{2}(\dot{y} - \dot{\phi}x)^2\right), \tag{55}$$

which is invariant under the gauge transformations generated by F [see Eq. (8)].

The set of variables (x,y,ϕ) is now enlarged by the introduction of the bosonic variable b and by the fermionic ghost pairs η and $\overline{\eta}$. For each of the variables $(x,y,\phi,b,\eta,\overline{\eta})$, an *antivariable* $(x^*,y^*,\phi^*,b^*,\eta^*,\overline{\eta}^*)$ is introduced. If the variable is a boson (fermion), the corresponding antivariable is a fermion (boson). Therefore, the number of additional variables has once again been drastically increased. The doubling of the variables allows for the definition of a bracket structure known as the *antibracket*. The antibracket plays an analogous role to the Poisson bracket, because within this structure variables are conjugate to antivariables. The antibracket of arbitrary functionals of the variables and antivariables is defined as

$$(A,B)\equiv\int d\tau\left(\frac{\delta A}{\delta q_s(\tau)}\frac{\delta^L B}{\delta q_s^*(\tau)}-\frac{\delta^R A}{\delta q_s^*(\tau)}\frac{\delta B}{\delta q_s(\tau)}.\right.$$
$$\left.+\frac{\delta^R A}{\delta \sigma_s(\tau)}\frac{\delta B}{\delta \sigma_s^*(\tau)}-\frac{\delta A}{\delta \sigma_s^*(\tau)}\frac{\delta^L B}{\delta \sigma_s(\tau)}\right), \tag{56}$$

where $q_s(\tau)$ and $\sigma_s(\tau)$ denote boson and fermion variables, respectively. The super-indices L and R, appearing in the functional derivatives with respect to the fermion variables, indicate differentiation to the left and to the right, respectively.[12]

Now, the so-called *master* equation,

$$(S,S)=0, \tag{57}$$

must be solved for a real boson functional of the variables $(x,y,\phi,b,\eta,\overline{\eta})$ and the antivariables $(x^*,y^*,\phi^*,b^*,\eta^*,\overline{\eta}^*)$, which becomes equal to the classical action S' when the antivariables vanish. It can be verified that the following functional[13,14]

$$S=S'+\int d\tau[\eta(xy^*-yx^*-\phi^*)-\overline{\eta}^*b] \tag{58}$$

is a solution to Eq. (57).

The action S is the starting point for quantizing the theory, for which a gauge fixing procedure must be implemented. With this aim we choose a function Ψ, an imaginary fermion functional of the variables, that is called the *gauge fixing fermion*. It plays a role analogous to that of ρ in Eqs. (23) and (48),

$$\Psi=i\int d\tau\left(\frac{1}{r_0}y-\ddot{\phi}+\frac{i}{2r_0^2}b\right)\overline{\eta}, \tag{59}$$

and we fix the value of the antivariables by the conditions

$$q_s^*=\frac{\delta\Psi}{\delta q_s},\quad \sigma_s^*=\frac{\delta\Psi}{\delta\sigma_s}. \tag{60}$$

We obtain

$$x^*=0,\quad \eta^*=0, \tag{61a}$$

$$y^*=\frac{i}{r_0}\overline{\eta},\quad \overline{\eta}^*=\frac{i}{r_0}y-i\ddot{\phi}-\frac{1}{2r_0^2}b, \tag{61b}$$

$$\phi^*=-i\ddot{\overline{\eta}},\quad b^*=-\frac{1}{2r_0^2}\overline{\eta}. \tag{61c}$$

If we substitute these expressions in Eq. (58), together with the value $x=r_0$, we derive the gauge fixed action for the motion along the ring, namely

$$S_\Psi=\int d\tau\left[\frac{1}{2}(\dot{y}-r_0\dot{\phi})^2+i\eta\overline{\eta}-i\eta\ddot{\overline{\eta}}-ib\left(\frac{1}{r_0}y-\ddot{\phi}\right)\right.$$
$$\left.+\frac{1}{2r_0^2}b^2\right]. \tag{62}$$

By completing squares and eliminating the variable b upon integration in the associated path integral, we find

$$S'_\Psi=\int d\tau\left[\frac{1}{2}(\dot{y}-r_0\dot{\phi})^2+\frac{1}{2}(y-r_0\ddot{\phi})^2-i\overline{\eta}\eta-i\dot{\overline{\eta}}\dot{\eta}\right]. \tag{63}$$

Although the Lagrange multiplier λ introduced in Eq. (11) has been treated as a degree of freedom independent of the collective coordinate ϕ, it may be related through the commutation relation

$$\dot{\phi}=i[(H-\lambda F),\phi]=i\lambda[I,\phi]=\lambda. \tag{64}$$

The inclusion of the nil term $(\lambda-\dot{\phi})I$ and the replacement of $\dot{\phi}$ by λ in Eq. (63) yields

$$S''_\Psi=S_{\mathrm{int}}+S_{\mathrm{coll}}+S_{\mathrm{coup}}, \tag{65a}$$

$$S_{\mathrm{int}}=\int d\tau\left[\frac{1}{2}(\dot{y}-r_0\lambda)^2+\frac{1}{2}(-r_0\dot{\lambda}+y)^2-i\dot{\overline{\eta}}\dot{\eta}-i\overline{\eta}\eta\right]$$
$$=\int d\tau\left[\frac{1}{2}(y^2+\dot{y}^2)+\frac{r_0^2}{2}(\lambda^2+\dot{\lambda}^2)\right.$$
$$\left.-r_0\frac{d(y\lambda)}{d\tau}-i\overline{\eta}\eta-i\dot{\overline{\eta}}\dot{\eta}\right]$$
$$=\int d\tau\left[\frac{1}{2}(y^2+\dot{y}^2)+\frac{r_0^2}{2}(\lambda^2+\dot{\lambda}^2)-i\overline{\eta}\eta-i\dot{\overline{\eta}}\dot{\eta}\right], \tag{65b}$$

$$S_{\mathrm{coll}}=-i\int d\tau\,\dot{\phi}I, \tag{65c}$$

$$S_{\mathrm{coup}}=i\int d\tau\,\lambda I. \tag{65d}$$

The action S_{int} for the intrinsic variables (original coordinate y, Lagrange multiplier λ and ghosts $\eta,\overline{\eta}$) contains the transformed action (6), but with the Lagrange multipliers as the velocities of the moving frame. Note the elimination of the total time derivative appearing in the second line. The action S_{int} also includes the gauge fixing terms and the action for the ghosts. The intrinsic system is analogous to the unphysical sector (29) displaying two boson and two fermion modes, all of which have the same excitation frequency one. The gauge fixing procedure [see Eqs. (59) and (60)] was chosen so as to give nonzero frequencies to these spurious modes.

S_{coll} is the free action for the collective coordinates in Hamiltonian form. From it we see that $I=\delta S_{\mathrm{coll}}/\delta\dot{\phi}$ is to be interpreted as the canonical collective momentum conjugate to the angle ϕ.

The coupling between collective and intrinsic degrees of freedom is given by S_{coup}.

VII. CONCLUSIONS

We have applied the sophisticated tools currently employed in the quantization of gauge theories to very simple mechanical models, both within the Hamiltonian and Lagrangian formulations. Such examples also illustrate the treatment of broken internal symmetries. Thus they are also useful in many-body finite systems (for example, molecular and nuclear) in which the concepts of symmetry transformations and broken symmetries are also relevant. They replace more conventional and cumbersome projection techniques.

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