Fermion-boson interactions and quantum algebras

A. Ballesteros,[1,*] O. Civitarese,[2,†] F. J. Herranz,[1,‡] and M. Reboiro,[1,2,§]
[1]*Departamento de Física, Universidad de Burgos, Pza. Misael Bañuelos, E-09001 Burgos, Spain*
[2]*Departamento de Física, Universidad Nacional de La Plata, c.c. 67 1900, La Plata, Argentine*
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Quantum algebras (q algebras) are used to describe interactions between fermions and bosons. Particularly, the concept of a $su_q(2)$ dynamical symmetry is invoked in order to reproduce the ground state properties of systems of fermions and bosons interacting via schematic forces. The structure of the proposed $su_q(2)$ Hamiltonians, and the meaning of the corresponding deformation parameters, are discussed.

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I. INTRODUCTION

Group-theoretical methods have contributed significantly to the study of the nuclear quantum-many-body problem. Classical examples are the Lipkin model [1], the Elliot su(3) model [2], the various realizations of Arima and Iachello Interacting Boson Model (IBM) [3], the Schütte and Da Providencia model [4], and the bi-fermion algebraic model of Geyer *et al.* [5], among other important contributions. We refer the reader to the review article of Klein and Marshalek [6], for a comprehensive presentation of the problem.

Models based on the coupling between bi-fermions and bosons have been introduced long ago [7,8]. These models are particularly suitable to describe condensation phenomena and transitions from Fermionic to Bosonic phases. In particular, the model proposed by Da Providencia and Schütte (DPS) is a solvable model which exhibits a phase transition between nucleonic and pionic condensates [4]. Similar ideas have been applied to mock up the fundamentals of the nonperturbative, low energy, regime of QCD [9,10]. Recently, an extension of the Lipkin (LE) model was proposed to take into account the interaction of pairs of bi-fermions (quark-antiquark pairs) with external bosons (gluon pairs) [11]. These constructions may help, as toy models, to understand interactions between fermions and bosons in hadron physics [12]. All these models share the nonperturbative nature of the approximations [13].

In addition to the conventional group theoretical approach, the literature is rich in different realizations of deformed algebras (q algebras) whose mathematical foundations can be found, i.e., in Refs. [14–17]. Applications of concepts related to q algebras to some selected quantum mechanical examples can be found in Ref. [18]. Recently published works on q algebras, in connection with quantum many-body Hamiltonians [19,20], have shown the suitability of the concept.

Several q-deformed versions of different schematic nuclear models have been previously introduced through the substitution of the symmetries of the original models by their q analogs (see Ref. [18] and references therein). An example of this ''direct q-deformation'' approach for a quantum optical fermion-boson Hamiltonian was given in Ref. [21]. The phonon spectrum in ^{4}He has been also described by using a q analog of a N-oscillator Hamiltonian in Ref. [22].

Although the mathematical meaning of the deformation parameter (q) which characterizes the commutation (anticommutation) relations between the generators of a given algebraic structure is self-evident [14–17], the physical meaning of it is less known. We recall that the exact q-deformed su(2) symmetry underlies the complete integrability of quantum systems like the *XXZ* Heisenberg chain [17,23–25] and Bloch electrons in a magnetic field [26]. In nuclear models, except for the quantum harmonic oscillator and the quantum rotor [18] where q may be interpreted as a stretching parameter of the corresponding spectra, little is known about the potentiality of q algebras in more realistic cases.

With the above motivation in mind, and as a complementary tool to standard nonperturbative many-body techniques [13], in this paper we propose the use of quantum algebras in the construction of effective Hamiltonians. We shall show that the quantum algebra $su_q(2)$ [27] can be used to define, in a natural way, new effective Hamiltonians which reproduce the same ground state properties and the spectrum of the ones based on fermion-boson interactions. Advances in the same direction have been achieved in quantum optics, where the interaction term of the Dicke model has been described through a $su_q(2)$ effective Hamiltonian [28].

We would like to stress that such a quantum algebra approach introduces a remarkable simplification of the models without any significant loss of physical content. Explicitly, we shall show how the DPS and LE Hamiltonians, originally defined on a $su(2)\oplus h_3$ Lie algebra, where su(2) is the algebra of quasispin fermion operators and the Heisenberg algebra h_3 accounts for the boson degrees of freedom, can be defined on the $su_q(2)$ algebra alone. It is found that the new effective $su_q(2)$ Hamiltonians reproduce accurately the physical properties of the $su(2)\oplus h_3$ models, provided the deformation parameter q is suitably fitted in terms of physical constraints.

We are confident that the present treatment can be successfully applied to describe other physical systems, where the effective motion is determined by the interaction between elementary fermion and boson degrees of freedom. The cor-

*Email address: angelb@ubu.es
†Email address: civitare@venus.fisica.unlp.edu.ar
‡Email address: fjherranz@ubu.es
§Email address: reboiro@venus.fisica.unlp.edu.ar

respondence between the spectra of interacting fermion-boson systems and effective q-deformed purely Fermionic systems, which we demonstrate in this work for some selected examples, may be a general feature common to other fermion $\otimes$ boson systems. Such is the case, for example, of nucleons interacting strongly *via* nuclear λ-pole fields, which may also be represented as free nucleons moving in a deformed central potential [29]. Similarly, the nonperturbative domain of QCD, which is the theory of the interactions between quark and gluons, may be viewed as an effective theory of confined fermions [11,12], without gluons. To summarize, this paper is devoted to the study of the equivalence between systems of interacting fermions and bosons and systems of q-deformed fermions. Clearly, we shall not deal with the transformation of a given Hamiltonian onto a q-deformed space, since this procedure leads, in general, to a completely different Hamiltonian.

The paper is organized as follows. In Sec. II we present the basic aspects of the fermion-boson interaction models considered in the work, and construct the associated deformed effective Hamiltonians. In Sec. III we discuss the behavior of the exact solutions obtained for the different Hamiltonians. Conclusions are drawn in Sec. IV.

II. FORMALISM

In this section we shall briefly review the essentials of the DPS and LE schematic models and discuss their realizations in the framework of deformed algebras (hereon referred to as q algebras).

A. DPS model

The DPS model [4] consists of $N=2\Omega$ fermions moving in two single shells. Each shell has a degeneracy 2Ω, and its substates are labeled by the index $l=1,\ldots,2\Omega$. The energy difference between shells is fixed by the energy scale ω_f. The creation and annihilation operators of particles belonging to the upper level, are denoted by $a_l^\dagger$ and a_l, respectively, while in the lower level, the creation and annihilation operators for holes are denoted by $b_l^\dagger$ and b_l. The fermions are coupled to an external boson field represented by the creation (annihilation) operators $B^\dagger$ (B) and by the energy ω_b, respectively. The DPS model Hamiltonian can be interpreted as the one describing a system of N fermions (either nucleons or quarks), belonging to an isospin-(flavor) multiplet and N spin projections (colors) in interaction with bosons (either pions or gluons), in a hadron (QCD) scenario. The DPS Hamiltonian reads [4]

$$H=\omega_f(T_0+\Omega)+\omega_b B^\dagger B+G(T_+B^\dagger+T_-B), \tag{1}$$

where G is the strength of the interaction in the particle-hole channel.

The particle (ν) and hole $(\overline{\nu})$ number operators are given by

$$\nu=\sum_l a_l^\dagger a_l, \quad \overline{\nu}=\sum_l b_l^\dagger b_l, \tag{2}$$

and the following bi-linear combinations of fermion operators

$$T_+=\sum_l a_l^\dagger b_l^\dagger, \quad T_-=(T_+)^\dagger,$$

$$T_0=\frac{1}{2}(\nu+\overline{\nu})-\Omega, \tag{3}$$

are the generators of the su(2) algebra:

$$[T_0,T_+]=T_+, \quad [T_0,T_-]=-T_-, \quad [T_+,T_-]=2T_0. \tag{4}$$

The Hamiltonian of Eq. (1) commutes with the operator

$$P=B^\dagger B-\frac{1}{2}(\nu+\overline{\nu})=B^\dagger B-(T_0+\Omega). \tag{5}$$

Therefore the matrix elements of H can be calculated in a basis labeled by the eigenvalues of the number operators for bosons and fermions, as shown in Ref. [4],

$$|m_\Omega,n\rangle=\sqrt{\frac{(\Omega-m_\Omega)!}{(m_\Omega+\Omega)!(2\Omega)!n!}}\,T_+^{\Omega+m_\Omega}\,(B^\dagger)^n|0\rangle. \tag{6}$$

In this basis the eigenvalues of P are given by

$$P|m_\Omega,n\rangle=(n-m_\Omega-\Omega)|m_\Omega,n\rangle. \tag{7}$$

In particular, we shall diagonalize H in the subspace spanned by the states $|m_\Omega,L+m_\Omega+\Omega\rangle\equiv|m_\Omega;L,\Omega\rangle$ which have a fixed eigenvalue L of P,

$$P|m_\Omega;L,\Omega\rangle=L|m_\Omega;L,\Omega\rangle. \tag{8}$$

In this subspace, the nonzero matrix elements of H are

$$\langle m_\Omega;L,\Omega|H|m_\Omega;L,\Omega\rangle=\omega_b L+(\omega_f+\omega_b)(\Omega+m_\Omega), \tag{9a}$$

$$\langle m_\Omega+1;L,\Omega|H|m_\Omega;L,\Omega\rangle = G\sqrt{(\Omega+m_\Omega+1)(\Omega-m_\Omega)(L+\Omega+m_\Omega+1)}. \tag{9b}$$

The dimension of the finite-dimensional subspace associated to each fixed eigenvalue L varies depending on the positive or the negative character of L. For $L\geqslant 0$ the quantum number m_Ω can take the values

$$m_\Omega=-\Omega,-\Omega+1,\ldots,\Omega, \tag{10}$$

and the Hilbert's subspace has dimension $2\Omega+1$. In the case $L<0$, the values that m_Ω can take are

$$m_\Omega=-L-\Omega,-L-\Omega+1,\ldots,\Omega, \tag{11}$$

and accordingly, the dimension of the Hilbert's subspace is $2\Omega+L+1$.

DPS and effective $\mathbf{su}_q(2)$ *Hamiltonians*

The quantum algebra $su_q(2)$ is a Hopf algebra deformation of su(2) [27] whose generators are $\tilde{T}_\pm$ and $\tilde{T}_0$, and obey the commutation rules

$$[\tilde{T}_0,\tilde{T}_\pm]=\pm\tilde{T}_\pm,\quad [\tilde{T}_+,\tilde{T}_-]=[2\tilde{T}_0]_q. \tag{12}$$

Here the q analog $[x]_q$ of a given object x (either a c number or an operator) is defined by

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}}=\frac{\sinh(zx)}{\sinh(z)}. \tag{13}$$

Throughout the paper we shall use alternatively q and z (where $q=e^z$) as the deformation parameter, furthermore we shall assume that q is real. Recall that the su(2) algebra of Eq. (4) is recovered from Eq. (12) in the limit $q\to 1$ $(z\to 0)$.

When q is not a root of unity, the irreducible representations of $su_q(2)$ are obtained as a straightforward generalization of those of su(2) [14,17]. Namely,

$$\tilde{T}_0|j,m\rangle=m\ \ |j,m\rangle,$$

$$\tilde{T}_+|j,m\rangle=\sqrt{[j+m+1]_q[j-m]_q}\ \ |j,m+1\rangle,$$

$$\tilde{T}_-|j,m\rangle=\sqrt{[j-m+1]_q[j+m]_q}\ \ |j,m-1\rangle. \tag{14}$$

The matrix elements of Eq. (9) correspond to a tridiagonal finite dimensional matrix. Let us consider an effective Hamiltonian, sharing the same property, which is defined as the following function of the $su_q(2)$ generators:

$$H_q=\omega_b L+(\omega_b+\omega_f)(\tilde{T}_0+\Omega)+\chi(q)q^{\tilde{T}_0/2}(\tilde{T}_++\tilde{T}_-)q^{\tilde{T}_0/2}, \tag{15}$$

where $\chi(q)$ is a scalar function and H_q will be realized in a $su_q(2)$ irreducible representation with the same dimension as the subspace spanned by $|m_\Omega;L,\Omega\rangle$ [therefore with $j=j(\Omega,L)$]. From Eq. (14), the nonvanishing matrix elements of Eq. (15) read

$$\langle j,m|H_q|j,m\rangle=\omega_b L+(\omega_f+\omega_b)(m+\Omega), \tag{16a}$$

$$\langle j,m+1|H_q|j,m\rangle=\chi(q)q^{(m+1/2)}\sqrt{[j+m+1]_q[j-m]_q}. \tag{16b}$$

In order to fit the dimensions, in the previous equations we take $j=\Omega$ and $m=m_\Omega$ for the effective $L\geqslant 0$ model, while for $L<0$, $j=\Omega+L/2$ and $m=m_\Omega+L/2$.

Note that, apparently, the Hamiltonians H of Eq. (1) and H_q of Eq. (15) seem to be quite different, since the latter has no Bosonic degrees of freedom. In fact, the nondeformed limit $q\to 1$ of Eq. (15) is the nondeformed su(2) Hamiltonian

$$H=\omega_b L+(\omega_b+\omega_f)(T_0+\Omega)+\chi(1)(T_++T_-). \tag{17}$$

which cannot be obtained from Eq. (1) through any transformation. The main result of this procedure is that the Bosonic degrees of freedom included in Eq. (1) may be absorbed by the q deformation in Eq. (15) provided that q is defined as an appropriate function of both Ω and L, a tradeoff leading to the purely Fermionic structure of Eq. (15). In this way it is possible to regard H_q as an effective Hamiltonian with physical properties similar to those of H. In particular, we shall determine numerically the optimal values of the deformation parameter q by imposing that the spectrum of the q Hamiltonian of Eq. (15) be as close as possible to that of Eq. (1). In so doing, the function $\chi(q)$ has been chosen as

$$\chi(q)=\frac{G\sqrt{L+\Omega+m_\Omega^0+1}\sqrt{(\Omega+m_\Omega^0+1)(\Omega-m_\Omega^0)}}{q^{(m^0+1/2)}\sqrt{[j+m^0+1]_q[j-m^0]_q}}, \tag{18}$$

where m_Ω^0 is the value of m_Ω that maximizes the matrix element of Eq. (9b), and $m^0=m_\Omega^0$ for $L\geqslant 0$, while $m^0=m_\Omega^0+L/2$ for $L<0$. This choice ensures that the maximum values of the interaction terms of the Hamiltonians H and H_q coincide (see Ref. [28]), as it is shown in Sec. III.

The main role of the exponentials $q^{\tilde{T}_0/2}$ in Eq. (15) is to break the $m\leftrightarrow -m$ symmetry of the effective model, since this is one of the main effects of the nonlinearity introduced by the fermion-boson coupling in Eq. (1). This effect could be reproduced, also, through functions others than exponentials of the $\tilde{T}_0$ operator. The effective fermionic Hamiltonian could also be defined by using more involved functions on the nondeformed su(2) algebra, since the main constrain is the block structure of Eq. (1). Nevertheless, we would like to stress that the essential advantage of using both the $su_q(2)$ operators of Eq. (14) and the exponential form of the effective Hamiltonian of Eq. (15) is that the eigenvalues of the interaction term

$$H_q^{\rm int}=q^{\tilde{T}_0/2}(\tilde{T}_++\tilde{T}_-)q^{\tilde{T}_0/2} \tag{19}$$

are just the q numbers $[2m]_q$ $(m=-j,\ldots,+j)$, and its eigenvectors are known in analytic form. They are related to q-Krawtchouk polynomials [28]. Therefore the results here presented seem to indicate that certain interactions between fermions and bosons can be accurately described by using q fermions as quasiparticles (i.e., effective Fermionic degrees of freedom) under the exactly solvable interaction given by the Hamiltonian $H_q^{\rm int}$.

B. Extended Lipkin models

As a second example of Hamiltonians including Fermionic and Bosonic degrees of freedom, let us introduce the Lipkin-type Hamiltonian

$$H=\omega_f(T_0+\Omega)+\omega_b B^\dagger B+G(T_+^2B+T_-^2B^\dagger). \tag{20}$$

The fermion sector of the model is described by two levels, with energies $\pm\omega_f/2$ and degeneracies 2Ω. The fermions interact with bosons of energy ω_b.

The Hamiltonian of Eq. (20) commutes with the operator [30]

$$P_{(+)}=B^{\dagger}B+\frac{1}{2}(T_0+\Omega), \tag{21}$$

therefore the matrix elements of H can be calculated in a basis labeled by the eigenvalues of P.

If we consider the basis of Eq. (6), we find that the eigenvalues of $P_{(+)}$ are given by

$$P_{(+)}|m_\Omega,n\rangle=\left(n+\frac{1}{2}(\Omega+m_\Omega)\right)|m_\Omega,n\rangle, \tag{22}$$

we shall consider the subspace spanned by the states with L fixed $|m_\Omega,L-\frac{1}{2}(\Omega+m_\Omega)\rangle\equiv|m_\Omega;L,\Omega\rangle$.

Hence the nonzero matrix elements of H read

$$\langle m_\Omega;L,\Omega|H|m_\Omega;L,\Omega\rangle=\omega_b L+\left(\omega_f-\frac{1}{2}\omega_b\right)(\Omega+m_\Omega), \tag{23a}$$

$$\langle m_\Omega+2;L,\Omega|H|m_\Omega;L,\Omega\rangle=G\sqrt{L-\frac{1}{2}(\Omega+m_\Omega)}\times\sqrt{(\Omega+m_\Omega+2)(\Omega+m_\Omega+1)(\Omega-m_\Omega)(\Omega-m_\Omega-1)}, \tag{23b}$$

with

$$\begin{aligned}
&L\geqslant\Omega, L \text{ integer}, \quad m_\Omega+\Omega=0,2,\ldots,2\Omega,\\
&L>\Omega, L \text{ half integer},\\
&m_\Omega+\Omega=1,3,\ldots,2\Omega-1,\\
&L<\Omega, L \text{ integer}, \quad m_\Omega+\Omega=0,2,\ldots,2L,\\
&L<\Omega, L \text{ half integer},\\
&m_\Omega+\Omega=0,2,\ldots,2L-1.
\end{aligned} \tag{24}$$

The Hamiltonian

$$H=\omega_f(T_0+\Omega)+\omega_b B^{\dagger}B+G(T_+^2B^{\dagger}+T_-^2B) \tag{25}$$

is another Lipkin-type Hamiltonian, which differs from Eq. (20) in the ground state correlations [11]. Since it commutes with the operator

$$P_{(-)}=B^{\dagger}B-\frac{1}{2}(T_0+\Omega), \tag{26}$$

its matrix elements can be calculated in a basis labeled by the eigenvalues of $P_{(-)}$. Once again this basis is just Eq. (6), where the eigenvalues of $P_{(-)}$ read

$$P_{(-)}|m_\Omega,n\rangle=\left(n-\frac{1}{2}(\Omega+m_\Omega)\right)|m_\Omega,n\rangle, \tag{27}$$

and we shall compute the matrix elements in the subspace spanned by the states $|m_\Omega,L+\frac{1}{2}(\Omega+m_\Omega)\rangle\equiv|m_\Omega;L,\Omega\rangle$. The nonzero matrix elements of H are given by

$$\langle m_\Omega;L,\Omega|H|m_\Omega;L,\Omega\rangle=\omega_b L+\left(\omega_f+\frac{1}{2}\omega_b\right)(\Omega+m_\Omega), \tag{28a}$$

$$\langle m_\Omega+2;L,\Omega|H|m_\Omega;L,\Omega\rangle=G\sqrt{L+\frac{1}{2}(\Omega+m_\Omega)+1}\times\sqrt{(\Omega+m_\Omega+2)(\Omega+m_\Omega+1)(\Omega-m_\Omega)(\Omega-m_\Omega-1)}, \tag{28b}$$

where the dimension of the subspace depends on L and Ω in the form

$$\begin{aligned}
&L\geqslant 0, \quad L \text{ integer}, \quad m_\Omega+\Omega=0,2,\ldots,2\Omega,\\
&L>0, \quad L \text{ half integer},\\
&m_\Omega+\Omega=1,3,\ldots,2\Omega-1,\\
&L<0, \quad L \text{ integer}, \quad m_\Omega+\Omega=-2L,-2L+2,\ldots,2\Omega,\\
&L<0, \quad L \text{ half integer},\\
&m_\Omega+\Omega=-2L,-2L+2,\ldots,2\Omega-1,
\end{aligned} \tag{29}$$

1. Extended Lipkin and $\mathbf{su}_q(2)$ *effective Hamiltonians*

As for the case of the DPS model, we introduce an effective Hamiltonian for Eq. (20), which is again defined on the $su_q(2)$ generators,

$$H_q=\omega_b L+\left(\omega_f-\frac{1}{2}\omega_b\right)(\tilde{T}_0+\Omega)+\chi(q)q^{\tilde{T}_0}(\tilde{T}_+^2+\tilde{T}_-^2)q^{\tilde{T}_0}. \tag{30}$$

This Hamiltonian has the following nonvanishing matrix elements

$$\langle j,m|H_q|j,m\rangle=\omega_b L+\left(\omega_f-\frac{1}{2}\omega_b\right)(\Omega+m), \tag{31a}$$

$$\langle j,m+2|H_q|j,m\rangle=\chi(q)q^{2(m+1)}\sqrt{[j+m+2]_q[j+m+1]_q[j-m]_q[j-m-1]_q}. \tag{31b}$$

[As the first step in order to fit the dimensions of Eq. (20) and of Eq. (30) we have to find the appropriate relation $j=j(\Omega,L)$ and, as a consequence, $m=m(m_\Omega,\Omega,L)$. Afterwards, we consider as the effective Hamiltonian the restriction of the matrix elements of Eq. (31a) to the invariant subspace spanned by $|j,m\rangle$ with $m=-j,-j+2,\ldots,j-2,j$. In this way we obtain the effective matrix elements

$$\langle j,m+2|H_q|j,m\rangle=\chi(q)h(L,\Omega,m_\Omega). \tag{32}$$

For values of $L\geqslant\Omega$ (L integer), we find $j=\Omega$, $m=m_\Omega$ and the function $h(L,\Omega,m_\Omega)$ is

$$h(L,\Omega,m_\Omega)=q^{2(m_\Omega+1)}\sqrt{[\Omega+m_\Omega+2]_q[\Omega+m_\Omega+1]_q[\Omega-m_\Omega]_q[\Omega-m_\Omega-1]_q}. \tag{33}$$

When $L<\Omega$ (L integer), we have that $j=L+\frac{1}{2}$, $m=m_\Omega+\Omega-L-\frac{1}{2}$, and

$$h(L,\Omega,m_\Omega)=q^{2(m_\Omega+\Omega-L+1/2)}\sqrt{[\Omega+m_\Omega-2L]_q[\Omega+m_\Omega-2L-1]_q}\,\sqrt{[\Omega+m_\Omega+1]_q[\Omega+m_\Omega+2]_q}. \tag{34}$$

The function $\chi(q)$ is defined by

$$\chi(q)=G\sqrt{L-\frac{1}{2}(\Omega+m_\Omega^0)}\frac{\sqrt{(\Omega+m_\Omega^0+2)(\Omega+m_\Omega^0+1)(\Omega-m_\Omega^0)(\Omega-m_\Omega^0-1)}}{h(L,\Omega,m_\Omega^0)}, \tag{35}$$

where m_Ω^0 is chosen as the value of m_Ω that maximizes Eq. (23b). Following the arguments presented in subsection A we shall search for values of the q-dependent coupling [of Eqs. (30) and Eq. (35)] which may absorb Bosonic degrees of freedom of Eq. (20) and yields a comparable spectrum for the purely Fermionic q-deformed Hamiltonian of Eq. (30).

Similarly, for the Hamiltonian of Eq. (25), we can write the effective $su_q(2)$ coupling

$$H_q=\omega_b L+\left(\omega_f+\frac{1}{2}\omega_b\right)(\tilde{T}_0+\Omega)+\chi(q)q^{\tilde{T}_0}(\tilde{T}_+^2+\tilde{T}_-^2)q^{\tilde{T}_0}. \tag{36}$$

Now the matrix elements of H_q are given by

$$\langle j,m|H_q|j,m\rangle=\omega_b L+\left(\omega_f+\frac{1}{2}\omega_b\right)(\Omega+m), \tag{37a}$$

$$\langle j,m+2|H_q|j,m\rangle=\chi(q)q^{2(m+1)}\sqrt{[j+m+2]_q[j+m+1]_q[j-m]_q[j-m-1]_q}. \tag{37b}$$

Note that the Hamiltonians of Eqs. (30) and (36) differ in the unperturbed sector, and that once again we have to fit the dimension of the $su_q(2)$ operator through the appropriate choice of the quantum numbers j and m. For values of $L\geqslant 0$ (L integer), we find $j=\Omega$ and $m=m_\Omega$. In the case $L<0$ (L integer), we have $j=L+\Omega+\frac{1}{2}$ and $m=m_\Omega+L+\frac{1}{2}$. We recall that the effective $su_q(2)$ matrix is given by the matrix elements of Eq. (37) computed within the subspace spanned by $|j,m\rangle$ where $m=-j,-j+2,\ldots,j-2,j$.

Finally, the adopted expression for $\chi(q)$, in Eq. (36) is

$$\chi(q)=Gq^{-2(m^0+1)}\sqrt{L+\frac{1}{2}(\Omega+m_\Omega^0)+1}\,\sqrt{\frac{(\Omega+m_\Omega^0+2)(\Omega+m_\Omega^0+1)(\Omega-m_\Omega^0)(\Omega-m_\Omega^0-1)}{[j+m^0+2]_q[j+m^0+1]_q[j-m^0]_q[j-m^0-1]_q}}, \tag{38}$$

and m_Ω^0 is chosen as the value of m_Ω that maximizes Eq. (28b), accordingly, $m^0=m_\Omega^0$ for $L\geqslant 0$, and $m^0=m_\Omega^0+L+\frac{1}{2}$ for $L<0$.

Before ending with this section, we shall summarize the main steps of the above formalism. We have written, for different fermion $\otimes$ boson Hamiltonians, q-deformed purely Fermionic Hamiltonians where the information about boson degrees of freedom is absorbed in the definition of the q-dependent strength $\chi(q)$. The actual value of $\chi(q)$ depends on the deformation parameter, which may be determined from the comparison between the spectra of the fermion $\otimes$ boson and q-deformed fermion Hamiltonians. We shall discuss the feasibility of this procedure in the next section.

III. RESULTS AND DISCUSSION

We have calculated the spectra of the Hamiltonians introduced in the previous section. The calculations have been performed by fixing the following set of parameters: $\omega_f=\omega_b=1$, in arbitrary units of energy, and for $N=2\Omega=30$ particles, unless stated. The parameter $x=G\sqrt{2\Omega/\omega_f\omega_b}$, was taken as the dimensionless coupling between fermions and bosons, for the case of Hamiltonian of Eq. (1), and it is defined as $x=G(4\Omega/\sqrt{\omega_f\omega_b})$, for the case of the Hamiltonians of Eqs. (20) and (25). As we shall discuss later on, actual values of x are indicative of the phase preferred by the system (in the sense of the dominance of the Fermionic or Bosonic degrees of freedom on the structure of the ground state) [30]. In general, we shall talk of a *normal* phase, of any of the fermion $\otimes$ boson Hamiltonians of the previous section, when the correlated ground state is the eigenstate of the symmetry operator P with the eigenvalue $L=0$. The denomination *deformed* phase will be assigned to cases where the correlated ground state is an eigenstate of P with eigenvalue $L\neq 0$. The Bosonic or Fermionic structure of the deformed phase is determined by the sign of L, following the corresponding definition of P.

Let us start with the DPS model. The coupling $x=0.5$ yields a normal solution of the DPS Hamiltonian. The value $x=1.5$ is consistent with a deformed solution of it. Figure 1, cases (a) and (c), shows the evolution of the ground state upon L. In the same figure we present the results of the q-deformed Hamiltonian corresponding to the DPS Hamiltonian. Figure 1, cases (b) and (d), shows the behavior of the deformed parameter $z=\ln(q)$, as a function of L, which reproduces the ground state energies of the insets (a) and (c). The values of z have been chosen so that the ground states energies of the DPS model and the ones of the $\mathrm{su}_q(2)$ effective model of Eq. (15) differ in less than 1%. Figure 2 shows the evolution of the values of L, z and $\chi(q)$, at the absolute ground state, for different values of the coupling constant x. Two different phases can be identified, depending on the value of x. The normal phase corresponds to values of $x\leqslant 1$, with $L=0$ and z almost constant, and the deformed phase corresponds to values of $x>1$, with values of $L>0$ and decreasing values of z.

Figure 3 displays the comparison between the matrix elements of the DPS model and the ones obtained with the effective Hamiltonian of Eq. (15). The scaling of Eq. (18) was performed as indicated in the text. These results support nicely the adopted procedure, since the agreement between both set of matrix elements is rather acceptable.

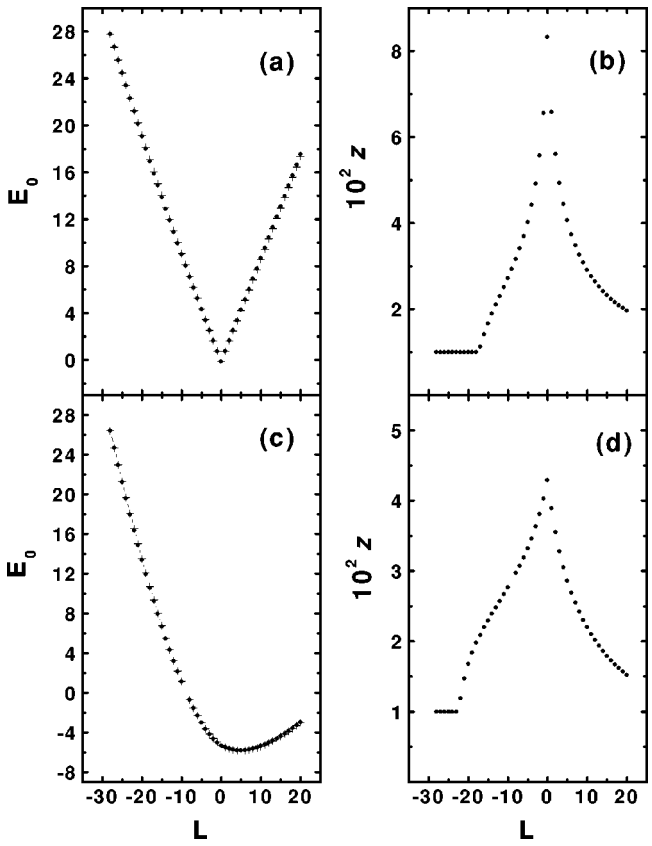

FIG. 1. Ground state energy E_0 in arbitrary units, and deformation parameter, $z=\ln(q)$, as a function of L. Insets (a) and (b) show results for the case $N=2\Omega=30$, $\omega_f=1$, $\omega_b=1$, and $x=G\sqrt{2\Omega/\omega_f\omega_b}=0.5$, while insets (c) and (d) correspond to $x=1.5$. The exact ground state energy corresponding to the DPS model of Eq. (1) is denoted by crosses while the one corresponding to the effective $\mathrm{su}_q(2)$ model of Eq. (15) is denoted by circles.

Figure 4 shows the results of the integrated Hamiltonian energy density, corresponding to the Hhamiltonias of Eqs. (1) and (15). Again in this case the agreement between both set of results was verified within the computer accuracy.

The above results, shown in Figs. 1–4, demonstrate that both the ground state energy and the spectrum of the DPS model can be represented by the effective $\mathrm{su}_q(2)$ Hamiltonian of Eq. (15), by fixing the value of $z(q)$, which is the parameter related with the q deformation.

A similar analysis can be performed for the LE models of Sec. II B. Figure 5 represents the ground state energy of the fermion $\otimes$ boson Hamiltonian of Eq. (20), and the behavior of the parameter z of the corresponding q-deformed version, Eq. (30). Also, in the same figure, the ground state energy of the q-deformed Hamiltonian [Eq. (30)] is given as a function of L. The insets (a) and (b) of Fig. 5 show the results corresponding to $x=0.5$ (normal phase), while insets (c) and (d) show the results obtained with $x=1.5$ (deformed phase). As for the case of the DPS model, we have chosen z so that the

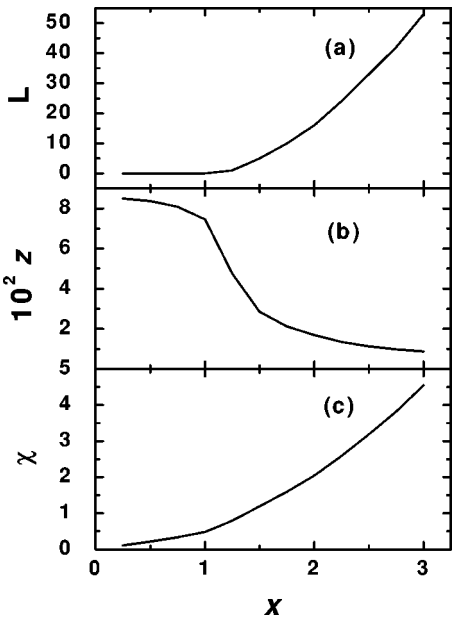

FIG. 2. Values of L [inset (a)], z [inset (b)], and χ [inset (c)], at the absolute ground state energy, as a function of x. The figure displays the results corresponding to the case $N=30$, $\omega_f=1$, and $\omega_b=1$ for different values of x, and for the Hamiltonians of Eqs. (1) and (15) (see Fig. 1).

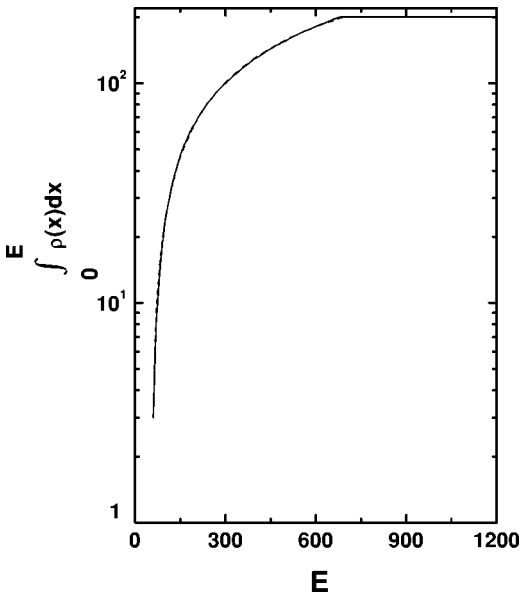

FIG. 4. Integrated density of states, as a function of the energy, corresponding to the Hamiltonians of Eq. (1) (solid line) and of Eq. (15) (dashed line). Both results coincide in the curve shown in the figure. The calculations were performed for $N=200$, $\omega_f=1$, $\omega_b=1$, and $x=1.5$. For the DPS model, Eq. (1), the value $L=34$ was used. The spectrum of the effective $su_q(2)$ Hamiltonian of Eq. (15) was calculated with $z=0.004$. Both curves coincide within the resolution of the diagram.

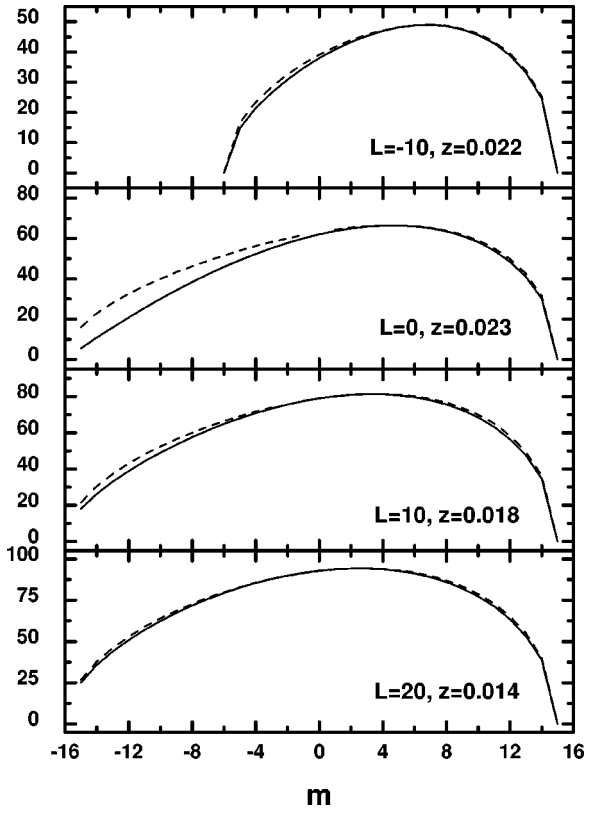

FIG. 3. Scaling procedure of Eq. (18). The matrix elements of Eq. (9b) (solid line) and Eq. (16b) (dashed line) are shown as a function of the m-quantum number. The values of L and z are indicated in the insets.

ground state energy of the Hamiltonian of Eq. (20) and that of Eq. (30) coincide within 1%, for each value of L.

Figure 6 displays the behavior of L, z, and $\chi(q)$, at the absolute ground state energy, for different values of the coupling constant x, for the Hamiltonians of Eqs. (20) and (30). As for the case of the DPS model, the results shown in this figure correspond to two different phases, which can be identified by the value of x. Similarly to the case of Fig. 2, the normal phase corresponds to $x \leqslant 1$, $L=0$, and $z(q)$ nearly constant. The deformed phase corresponds to $x>1$, $L\neq 0$, and increasing values of $z(q)$.

Figure 7 displays the comparison between the spectrum of the Hamiltonian of Eq. (20) and the spectrum of the effective Hamiltonian of Eq. (30). As done for the cases of Eqs. (1) and (15) (see Fig. 4) we have calculated the integrated Hamiltonian energy density (number of eigenvalues per unit energy interval). Also in this case, the scaling procedure yields almost identical results, within computer accuracy, as compared to the original Hamiltonian.

Finally, the ground state energies, the q-deformation parameter, the q-depending coupling, and the comparison between the spectra, for the case of the Hamiltonians of Eqs. (25) and (36), are shown in Figs. 8–10, respectively. From the results shown in Fig. 8, cases (b) and (d), it is seen that there is a particular value of L, for which $z(q)\approx 0$. It means that, for this particular value of $z(q)$, the su(2) symmetry is dynamically restored. Figure 9 shows the behavior of L [inset (a)], $z(q)$ [inset (b)] and $\chi(q)$ [inset (c)], taken at the absolute ground state energy, as a function of x. Figure 10 displays the comparison between the spectrum of the Hamil-

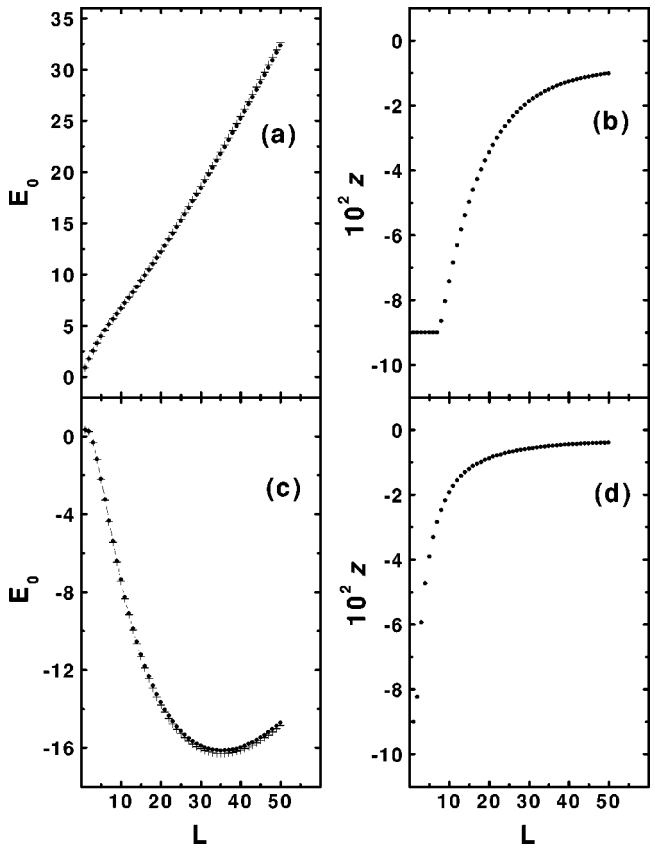

FIG. 5. Ground state energies and $z=\ln(q)$ for the LE model of Eq. (20) are shown as a function of L. Insets (a) and (b) show results for the case $N=2\Omega=30$, $\omega_f=1$ MeV, $\omega_b=1$ MeV, and $x=G(4\Omega/\sqrt{\omega_f\omega_b})=0.5$, while insets (c) and (d) correspond to $x=1.5$. The exact ground state energies corresponding to the LE model of Eq. (20) are denoted by crosses while the one corresponding to its associated effective $su_q(2)$ Hamiltonian of Eq. (30) are denoted by circles.

tonian of Eq. (25) and the one obtained with the effective Hamiltonian of Eq. (36), with $z=0.00440$.

A systematic feature emerges from the above discussed series of results and it is related with the replacement of the boson degrees of freedom, which are present in the considered initial Hamiltonians, by the effective q-dependent coupling. In the three cases which we have considered, the spectrum of the fermion $\otimes$ boson system and the spectrum of the q-deformed purely fermionic system agree, for certain nontrivial values of the q-deformation parameter $z(q)$. The procedure works reasonably, for the rotorlike structure of the DPS Hamiltonian, as well as for the vibrational-like structure of the LE Hamiltonians. There is a trend in the dependence of $z(q)$ upon L, which is the parameter associated to the symmetry in the fermion $\otimes$ boson space. It is symmetric for the case of the DPS Hamiltonian and almost asymmetric for the case of the Lipkin Hamiltonians. Also, $z(q)$ resembles more the behavior of an order parameter for the case of the Lipkin Hamiltonians than for the DPS one. Concerning relatives values of $z(q)$, the q-deformed versions of the Hamiltonians of Eqs. (1) and (20) required values of $z(q)$ (≈ 0.03) which are larger than the value of $z(q)$ correspond-

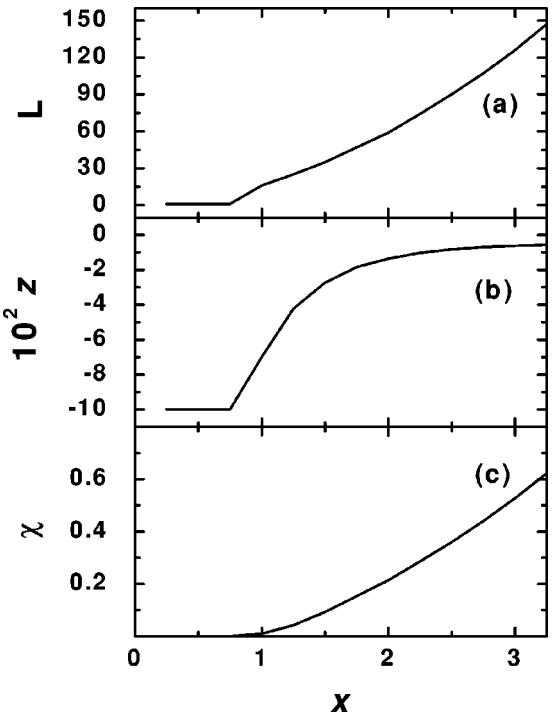

FIG. 6. Idem as Fig. 2, for the LE model of Eq. (20), and for the $su_q(2)$ Hamiltonian of Eq. (30). The values of the parameters N, ω_f, and ω_b are the same as those given in the captions to Fig. 5.

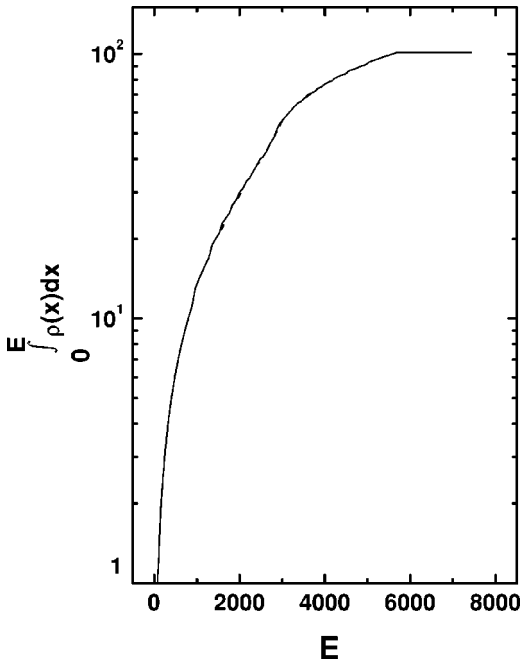

FIG. 7. Integrated density of states for the LE model of Eq. (20) (solid line) and for the effective $su_q(2)$ Hamiltonian of Eq. (30) (dashed line). The values of N, ω_f, and ω_b are the same as those given in the caption to Fig. 4. The results corresponding to the Hamiltonian of Eq. (20) have been obtained with $L/2\Omega=7$. For the effective $su_q(2)$ Hamiltonian of Eq. (30) the value $z=-0.0009$. Both curves coincide within the resolution of the figure.

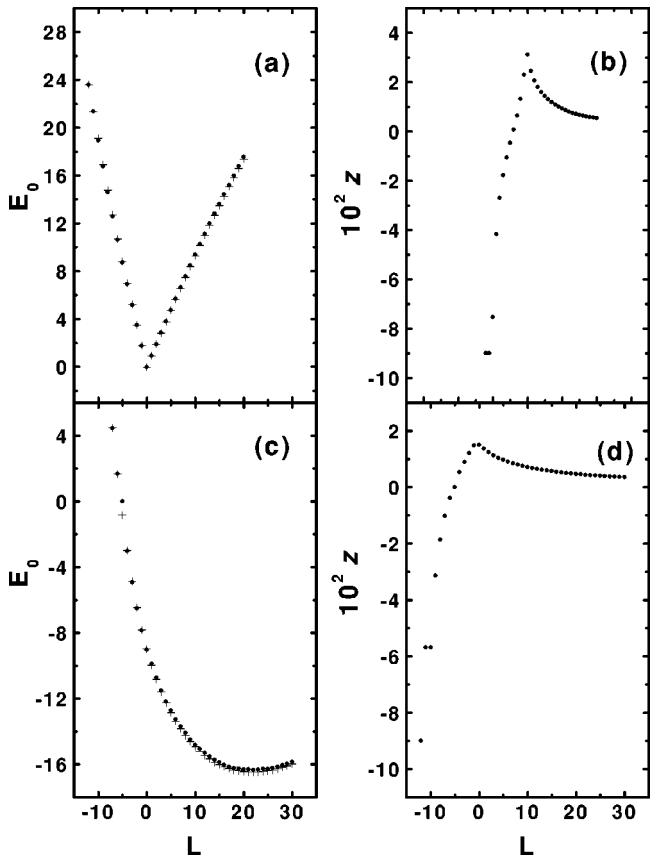

FIG. 8. Ground state energies and $z=\ln(q)$ for the LE model of Eq. (25), are shown as a function of L. Insets (a) and (b) show results for the case $N=2\Omega=30$, $\omega_f=1$ MeV, $\omega_b=1$ MeV, and $x=G(4\Omega/\sqrt{\omega_f\omega_b})=0.5$, while insets (c) and (d) correspond to $x=1.5$. The exact ground state energies corresponding to the LE model of Eq. (25) are denoted by crosses while the one corresponding to the associated effective $su_q(2)$ Hamiltonian of Eq. (36) are denoted by circles.

ing to the q-deformed version of Hamiltonian of Eq. (25). This result shows the sensitivity of the chosen value of z upon the vibrational or rotational-like character of the fermion $\otimes$ boson picture.

IV. CONCLUSONS

In this work we have shown that effective $su_q(2)$ Hamiltonians can be introduced in order to reproduce the ground state properties and the spectrum of different interacting fermion-boson Hamiltonians. In this respect, the Bosonic part of the interactions can be effectively embedded as an appropriate q deformation of the su(2) Fermionic algebra.

The results presented at this work show the existence of a close relation between the deformation parameter, $z(q)$, which fixes the strength $\chi(q)$ of the purely Fermionic q-deformed Hamiltonians, and the eigenvalue L of the symmetry operator P, associated to the fermion $\otimes$ boson Hamiltonians. Both $z(q)$, in the case of the $su_q(2)$ effective models, and L, for the fermion-boson interactions, display a critical behavior as a function of the coupling constant x.

Because of the relevance of the DPS model in the description of hadronic systems [31] and the nice agreement obtained with the q-deformed version of it, we are confident about the potentiality of q-deformed representations in more involved physical scenarios.

Work is in progress concerning the extension of the presented formalism to nonperturbative QCD.

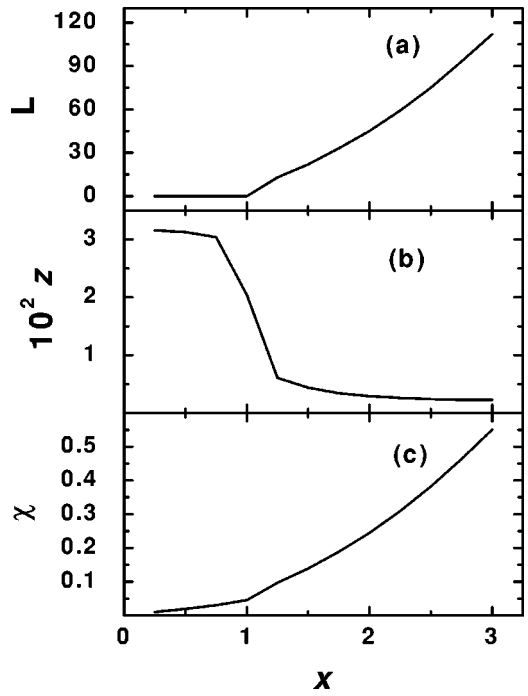

FIG. 9. Idem as Fig. 2, for the LE model of Eq. (25) and its correspondent q analog of Eq. (36). The values of N, ω_f, and ω_b are the ones given in the captions to Fig. 8.

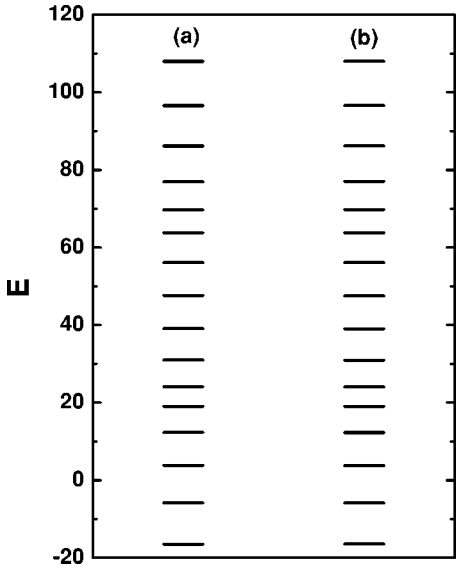

FIG. 10. The spectrum for the LE model of Eq. (25) and for the $su_q(2)$ Hamiltonian of Eq. (36). The values of N, ω_f, and ω_b are the same as those of Fig. 8. The spectrum denoted by (a) corresponds to the one obtained from the Hamiltonian of Eq. (25), for $L=22$. The spectrum denoted by (b) is obtained from the effective $su_q(2)$ Hamiltonian of Eq. (36), with $z=0.0044$.

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