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Fermion and boson condensates in a QCD-inspired model Hamiltonian

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The interaction between bi-fermionic and bosonic degrees of freedom is described by using an effective Hamiltonian inspired in QCD. The Dyson boson mapping technique is utilized in conjunction with a coherent state basis to construct energy landscapes in a multiparametric space. The appearance of fermionic and bosonic condensates is studied by minimizing the potential energy surface. The relationship between the domains of fermion and boson condensates and the order parameters of the model is discussed.

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I. INTRODUCTION

The use of bi-fermion operators in the context of the nuclear many-body problem has been advanced by Geyer [1] and Geyer *et al.* [2,3] and by other authors [4–6]. A model which includes the coupling between bi-fermions and bosons, and which closely resembles low energy QCD, was introduced by Schütte and Da Providencia [7]. The same model was applied to describe the coupling between effective degrees of freedom by other authors [8]. The inclusion of bi-fermion degrees of freedom in nonperturbative treatments of QCD has been introduced in [9]. The same type of model for bi-fermions has been investigated in systems describing high temperature superconductivity [10], i.e., in a completely different context. Concerning nonperturbative approaches to QCD, the recent literature provides definite examples about the use of diquarks [11] as the relevant degrees of freedom as well as the use of pairs of particle and/or hole excitations associated to generalized pions in dense QCD [12]. The role of diquarks and quarks in a model with confinement was studied in [13] while the weak-coupling limit in color superconductivity, by using pairs of quarks which are breaking the color symmetry, was reported in [14]. The gluon sector of QCD, treated by using boson mapping techniques, was studied in [15]. Finally, and concerning the use of boson expansion techniques in conventional field theory, the reader is kindly referred to the work of [16].

The finding of fermion and boson condensates, for the case of an effective Hamiltonian which contains pairs of fermions interacting with an external boson, was reported by Schütte and Da Providencia [7]. The Becchi, Rouet, Stora, and Tyutin (BRST) treatment of the same Hamiltonian can be found in [17]. The group structure of a purely bi-fermion Hamiltonian was investigated in a series of papers by Geyer *et al.* [2]. The extension of this Hamiltonian to a generalized bi-fermion $\otimes$ boson space has been proposed recently [18]. The most simple way of looking at condensates in systems where bi-fermion and boson operators are coupled is the one advanced by Schütte and Da Providencia [7], which consists of a mean field approximation followed by a variation which determines the fermionic or bosonic nature of the condensate via an order parameter. In the model of Schütte and Da Providencia [7], the order parameter is proportional to the vacuum expectation value of the bi-fermion or boson fields. These steps can also be followed when a more general bi-fermion-boson interaction is used, as the one of Ref. [18]. The use of a generalized definition of coherent states [19] as a tool to identify fermion and boson condensates has been advanced in [20] and in [21]. In this work we shall discuss the use of coherent states in the framework of a QCD-inspired Hamiltonian [18,20–22].

We shall introduce bi-fermion operators which represent quark-quark and quark-antiquark pairs [or particle-particle (pp), hole-hole (hh), and particle-hole (ph) pairs in the standard language of the nonrelativistic quantum many-body theory] and bosons (i.e., gluons or scalar and vector boson fields) [22]. We shall then map the operators by using Dyson's expansion method [23]. The expectation value of the transformed Hamiltonian will be calculated in a basis of coherent states. We have chosen a basis of coherent states because it is particularly suitable for the calculation of matrix elements of the transformed Hamiltonian. These matrix elements are expressed as functionals of order parameters, which are given in terms of vacuum expectation values of bi-fermion and boson operators, as done in [24].

The identification of bi-fermion and boson condensates is done by performing variations in a multiple-parametric space where the quantities to be determined are the vacuum expectation values of the bi-fermion and boson operators expanded in the basis of coherent states. As a test of consistency of the approximation, we have worked out a limiting case of our Hamiltonian which reproduces the result of the work of Schütte and Da Providencia [7].

Since the exact results of the model of Ref. [7] are known, we shall compare them with the results of the boson expansion. This is done in order to illustrate the meaning of the effective degrees of freedom based on bi-fermions and its potential use in models where the exact solution is unknown, as is the case of the QCD inspired Hamiltonians [20–22]. In this respect, the boson mapping of the bi-fermion and boson degrees of freedom in conjunction with the use of a basis of

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coherent states will provide us with a nonperturbative description of the Hamiltonian. The scope of this treatment will, as a matter of fact, be assessed by the above-mentioned comparison with the SO(3) limit of [7] and with the comparison between variational and exact results, for some particular cases where the diagonalization of the full Hamiltonian is feasible. It is well known that, in dealing with boson mapping techniques, the presence of spurious states can affect the results [2]. This problem has been studied in detail also by other authors [5,6] and it is still a matter of discussions [25]. In the present work, we are not concerned about the effects due to spurious states upon the complete set of solutions of the Hamiltonian. Rather, we shall focus on the validity of the variational treatment of the ground state and show to which extent spurious states are affecting our final results for the energy of the fermionic and bosonic condensates. For the particular case of the use of coherent states, as trial states, the effects due to the corresponding spurious states will be discussed in connection with the exact diagonalization of the fully mapped Hamiltonian in the subspace of functions with a definite P symmetry. In Sec. II we introduce the model and in Sec. III we present and discuss the results of the calculations. Conclusions are drawn in Sec. IV.

II. FORMALISM

To start with, we shall introduce an effective model which includes fermion and boson degrees of freedom [18]. The elementary building blocks are two fermion levels and three types of bosons associated to bi-fermion excitations. The fermions in the upper energy level represent particles and those in the lower energy level represent holes. The Hamiltonian is written as

$$H = H_0 + H_{ph} + H_{pp} + H_{hh}, \tag{1}$$

with

$$\begin{aligned} H_0 &= H_{0f} + H_{0b}, \\ H_{0f} &= \frac{\omega_F}{2}(\nu + \overline{\nu}), \\ H_{0b} &= \omega_b B_b^\dagger B_b + \omega_p B_p^\dagger B_p + \omega_h B_h^\dagger B_h, \\ H_{ph} &= G_1(S_+ B_b^\dagger + S_- B_b), \\ H_{pp} &= G_2(L_+ B_h^\dagger + L_- B_h), \\ H_{hh} &= G_3(K_+ B_p^\dagger + K_- B_p). \end{aligned} \tag{2}$$

The operators which appear in this equation are the boson operators representing particle-hole ($B_b^\dagger$), particle-particle ($B_p^\dagger$), and hole-hole ($B_h^\dagger$) excitations and the corresponding bi-fermion operators $S_\pm$, $L_\pm$, and $K_\pm$, respectively; ν and $\overline{\nu}$ are the fermion number operators of the upper and lower level, ω_F is the energy spacing between levels, and ω_b, ω_p, and ω_h are the energy of the bosons. $G_{(i=1,2,3)}$ are coupling constants, in units of energy, of the bi-fermion-boson couplings allowed by the model. Details on the notation are given in [1,18].[1] In this model, fermions and bosons are independent degrees of freedom while bi-fermion creation and annihilation operators are coupled to bosons in such a manner that the total number of fermions is conserved. The bosonic degrees of freedom describe external excitations which change the number of fermions, Δn_F, in 0 and ± 2, for particle-hole, particle-particle, and hole-hole bosons, respectively. In this respect we can think of the bosons as scalars if $\Delta n_F = 0$ and vectors if $\Delta n_F = \pm 2$.

A. Dyson mapping procedure and the variational solution of the model of Schütte and Da Providencia

We shall now illustrate the Dyson mapping of the Hamiltonian (1). For simplicity, we shall first work with the particle-hole channels of Eq. (1), by setting $G_2 = G_3 = 0$. In this limit the Hamiltonian of Eq. (1) coincides with the one discussed by Schütte and Da Providencia [7]. The Dyson boson mapping is performed by introducing the following relationships:

$$\begin{aligned} (S_+)_D &= b_F^\dagger (2\Omega - n_F), \\ (S_-)_D &= b_F, \\ (\nu)_D &= n_F, \end{aligned} \tag{3}$$

where $(\ )_D$ indicates the Dyson mapping, $n_F = b_F^\dagger b_F$ is the number of ph pairs, and $N = 2\Omega$ is the total number of fermions, Ω being half the degeneracy of each fermion level. After applying this transformation to the Hamiltonian, one gets

$$(H)_D = \omega_F n_F + \omega_b B_b^\dagger B_b + G_1 b_F^\dagger (2\Omega - n_F) B_b^\dagger + G_1 b_F B_b. \tag{4}$$

Next, we shall calculate the expectation value of the transformed Hamiltonian with respect to a trial state. The trial state should be defined in the most general form, in such a way that it includes all possible physically allowed states. In addition, we shall assume that the trial state can be parametrized by one (or several) order parameter(s). One possible choice is the use of a coherent state [19,26]. The coherent state is a product of a boson part and a fermion part. The boson part is given by the exponential of the boson creation operator multiplied by a complex order parameter. The fermion part of the trial state must obey the Pauli principle, i.e., we choose a coherent state which converges to an exponential in the limit of infinite degeneracy of the fermion levels ($\Omega \rightarrow \infty$). Also, we have to take into account the fact that the Dyson mapping is non-Hermitian, e.g., bra and ket states have different forms. The trial state for bosons is given by

$$|\alpha_{cb}\rangle = e^{\alpha_{cb} B_b^\dagger}|0\rangle, \tag{5}$$

[1] In this paper we follow closely the notation of Ref. [1]; in Ref. [18] the definitions of S and T are interchanged.

where $|0\rangle$ is the boson vacuum and $\alpha_{cb}=\alpha_b e^{i\phi_b}$ is the order parameter. For the fermion sector we propose the following trial states:

$$\langle\chi_{cF}|=\langle 0|\sum_{n=0}^{2\Omega}\frac{\chi_{cF}^{n*}}{n!}(S_-)_D^n,$$

$$|\chi_{cF}\rangle=\sum_{n=0}^{2\Omega}\frac{\chi_{cF}^{n}}{n!}(S_+)_D^n|0\rangle, \tag{6}$$

where $|0\rangle$ is the fermion vacuum and $\chi_{cF}=\chi_F e^{-i\xi_F}$ is the complex order parameter corresponding to the fermionic part. The sum of the previous equations is limited to $n_{\max}=2\Omega$, because the operator S_+ creates a particle-hole and 2Ω is the maximum number of ph pairs allowed by the Pauli principle.

When the Dyson boson mapping is used, the bra and ket states are not Hermitian conjugate but rather dual to each other.

By replacing in Eqs. (6) the fermion operators by their Dyson boson images, Eq. (3), one gets

$$\langle\chi_{cF}|=\langle 0|\sum_{n=0}^{2\Omega}\frac{\chi_{cF}^{n*}}{n!}(b_F)^n,$$

$$|\chi_{cF}\rangle=\sum_{n=0}^{2\Omega}\frac{\chi_{cF}^{n}}{n!}\prod_{k=1}^{n}(2\Omega-k+1)(b_F^\dagger)^n|0\rangle \tag{7}$$

(by definition, the factor in front of the boson creation operator is equal to 1 for $n=0$). The total trial state, which is not normalized, is given by the product of Eqs. (5) and (6), i.e.,

$$|\alpha_{cb}\chi_{cF}\rangle=|\alpha_{cb}\rangle|\chi_{cF}\rangle. \tag{8}$$

With these trial functions, the boson part of an arbitrary matrix element takes the form

$$\frac{\langle\alpha_{cb}|(B_b^\dagger)^{n_1}(B_b)^{n_2}|\alpha_{cb}\rangle}{\langle\alpha_{cb}|\alpha_{cb}\rangle}=(\alpha_b)^{(n_1+n_2)}e^{-i(n_1-n_2)\phi_b}, \tag{9}$$

and the fermionic part can be written

$$\begin{aligned}\frac{\langle\chi_{cF}|(b_F^\dagger)^{n_1}(b_F)^{n_2}|\chi_{cF}\rangle}{\langle\chi_{cF}|\chi_{cF}\rangle}&=\chi_F^{n_1+n_2}\sum_{n=0}^{\min(2\Omega-n_2,2\Omega-n_1)}e^{-i(n_1-n_2)\xi_F}\frac{\chi_F^{2n}}{n!}\\&\quad\times\prod_{k=1}^{n+n_2}(2\Omega-k+1)\\&=F_{n_1n_2}\chi_F^{n1+n2},\end{aligned} \tag{10}$$

where we have introduced the notation $F_{n_1n_2}\equiv F_{n_1n_2}(\chi_F,\xi_F)$ to represent the sum which appears in Eq. (10). The expectation value of the Hamiltonian (4), divided by the norm $\langle\alpha_{cb}\chi_{cF}|\alpha_{cb}\chi_{cF}\rangle=F_{00}$, is given by

$$\begin{aligned}E(\chi_F,\phi_b,\xi_F)=\omega_F\chi_F^2\frac{F_{11}}{F_{00}}+\omega_b\alpha_b^2\\+G_1N\chi_F\alpha_b\frac{F_{10}}{F_{00}}e^{-i(\phi_b+\xi_F)}\\-G_1\chi_F^3\alpha_b\frac{F_{21}}{F_{00}}e^{i(\phi_b+\xi_F)}\\+G_1\chi_F\alpha_b\frac{F_{01}}{F_{00}}e^{i(\phi_b+\xi_F)}.\end{aligned} \tag{11}$$

Real values of $E(\chi_F,\phi_b,\xi_F)$ are obtained for $\phi_b+\xi_F=0$ and π. The structure of the real part of the energy surface is of the form

$$A(\alpha_b,\chi_F)+B(\alpha_b,\chi_F)\cos(\phi_b+\xi_F), \tag{12}$$

which has a minimum at $\phi_b+\xi_F=\pi$, provided that $G_1>0$. Thus, it suffices to fix $\phi_b=\pi$ and $\xi_F=0$, since all values of ϕ_b and ξ_F such that $\phi_b+\xi_F=\pi$ yield the same minimum of the potential energy surface (PES).

In the previous equations we have applied the complete Dyson mapping. Another possibility is to perform the mapping by keeping the lowest order terms in the bosons. As long as the Hamiltonian is mapped exactly, the functional form of the trial state can be chosen quite arbitrarily without loss of generality. Even if the trial state is an exponential function, phase transitions can be identified as long as $N\leqslant\Omega$. For higher occupations the Pauli principle leads to instabilities and the potential energy does not have a lower limit. By keeping the lowest order terms in the boson mapping, approximately obeying the Pauli principle and by introducing a cutoff in the sum as explained above, the obtained PES saturates at a value which is not significantly different from the exact result. The exact solution of the Schütte and Da Providencia model is constructed as shown in [18] in the basis of the symmetry operator $P=B_b^\dagger B_b-\frac{1}{2}(\nu+\overline{\nu})$. Figure 1 shows the results of the exact calculation and those of the approximated one constructed at lowest order in the boson expansion of the trial state and obeying the Pauli principle approximately. These calculations have been performed for $\Omega=15$ and two different sets of coupling constants, as given in the caption to Fig. 1. As it is shown in Figs. 1(a) and 1(b), the exact and approximated PES agree quite well. It means that the truncation of the boson expansion, in the trial state, does not introduce significant differences, provided the Hamiltonian is mapped exactly.

B. Dyson boson mapping and the variational treatment of the complete Hamiltonian

The bi-fermion operators which appear in the Hamiltonian of Eq. (1) are part of the generators of the algebra of the $O(5)\simeq Sp(4,R)$ group [27]. The Dyson mapping of these

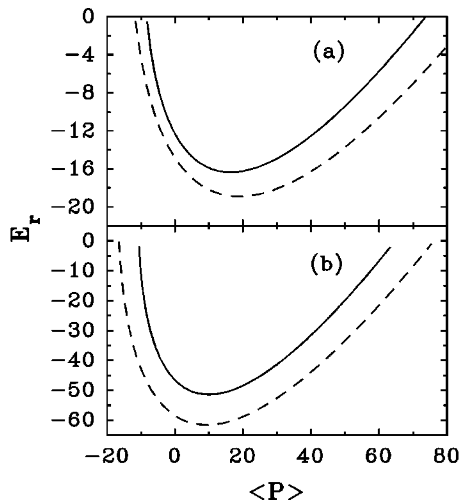

FIG. 1. Real part of the potential energy surface, E_r, in units of GeV, as a function of the expectation value of the symmetry operator P, for the case $G_1 \neq 0$, $G_2 = G_3 = 0$ [limit corresponding to the model of Schütte and Da Providencia (Ref. [3])]. With dashed and solid lines we indicate the approximated solution (Dyson mapping at lowest order in the trial wave function, all orders in the Hamiltonian), and the exact one, as described in the text (Sec. II A). The results shown in (a) correspond to $G_1 = 0.365$ GeV, $\Omega = 15$, and $\omega_F = \omega_b = 1$ GeV, while those of (b) correspond to $G_1 = 0.949$ GeV, $\Omega = 15$ and $\omega_F = 1$ GeV, $\omega_b = 3$ GeV.

operators is given in Ref. [1] and for completeness we have summarized the procedure in the following set of expressions:

$$
\begin{aligned}
(S_+)_D &= b_F^\dagger(2\Omega - n_F - 2n_P - 2n_H) - b_P^\dagger b_H^\dagger b_F, \\
(S_0)_D &= n_F + n_P + n_H - \Omega, \\
(S_-)_D &= b_F, \\
(T_+)_D &= 2b_F^\dagger b_H + b_P^\dagger b_F, \\
(T_0)_D &= b_P^\dagger b_P - b_H^\dagger b_H, \\
(T_-)_D &= 2b_F^\dagger b_P + b_H^\dagger b_F, \\
(L_+)_D &= b_P^\dagger(\Omega - n_P - n_F) - (b_F^\dagger)^2 b_H, \\
(L_0)_D &= n_P + \frac{1}{2}n_F - \frac{1}{2}\Omega, \\
(L_-)_D &= b_P, \\
(K_+)_D &= b_H^\dagger(\Omega - n_H - n_F) - (b_F^\dagger)^2 b_P, \\
(K_0)_D &= n_H + \frac{1}{2}n_F - \frac{1}{2}\Omega, \\
(K_-)_D &= b_H,
\end{aligned}
\tag{13}
$$

where n_F, n_P, and n_H are the number operators corresponding to the phonon (ph), pairon (pp), and holon (hh) pairs. Since the total number of fermions is conserved, the following relations are preserved by the mapping:

$$
\begin{aligned}
\nu &\rightarrow 2n_P + n_F, \\
\overline{\nu} &\rightarrow 2n_H + n_F.
\end{aligned}
\tag{14}
$$

We will now construct a coherent state which will be used to investigate phase transitions that may occur in the model defined by the Hamiltonian of Eq. (1). The fermion content of the bi-fermion operators, particle-hole, particle-particle, and hole-hole will be denoted by the subindexes F, P, and H, respectively, and lower case subindexes b, p, and h will be used to denote bosonic excitations. The trial state is represented by the coherent state, which is a product of a boson and a fermion part,

$$|\alpha,\chi\rangle = |\alpha_{cb}, \alpha_{cp}, \alpha_{ch}\rangle |\chi_{cF}, \chi_{cP}, \chi_{cH}\rangle, \tag{15}$$

with complex order parameters

$$\alpha_{ck} = \alpha_k e^{-i\phi_k}, \quad \chi_{cL} = \chi_L e^{-i\xi_L}, \tag{16}$$

with $k = b,p,h$ and $L = F,P,H$. The first factor on the right-hand side of Eq. (15) is the non-normalized boson coherent state given by

$$|\alpha\rangle = \exp(\alpha_{cb}B_b^\dagger + \alpha_{cp}B_p^\dagger + \alpha_{ch}B_h^\dagger)|0\rangle. \tag{17}$$

The fermionic coherent state is given by

$$
\begin{aligned}
|\chi\rangle = &\sum_{n_F=0}^{2\Omega} \sum_{n_P=0}^{\Omega-[n_F+1/2]} \sum_{n_H=0}^{\Omega-[n_F+1/2]} \\
&\times (\delta_1 \chi_{cF})^{n_F} (\delta_2 \chi_{cP})^{n_P} \\
&\times (\delta_2 \chi_{cH})^{n_H} \frac{(b_F^\dagger)^{n_F}}{n_F!} \frac{(b_P^\dagger)^{n_P}}{n_P!} \frac{(b_H^\dagger)^{n_H}}{n_H!} |0\rangle,
\end{aligned}
\tag{18}
$$

with $\delta_1 = 2\Omega$, $\delta_2 = \Omega$ for the ket state, and $\delta_1 = \delta_2 = 1$ for the bra state, and $[n_F + 1/2] = n_F/2$ for $n_F =$ even and $[n_F + 1/2] = n_F + 1/2$ for $n_F =$ odd. The ansatz in Eq. (18) corresponds to the lowest order mapping of the operator S_+ in the Dyson mapping. As shown in the preceding section, we can still describe phase transitions with this trial state, provided the Hamiltonian is mapped exactly. The upper limits in Eq. (18), represented as 2Ω and $\Omega - [n_F]$, are chosen such that the number of fermions in each level does not exceed 2Ω, a condition which is imposed by the partial fulfillment of the Pauli principle.

C. Spurious states

A frequently found drawback of the Dyson boson mapping of a fermion Hamiltonian is the ocurrence of spurious states, as pointed out by Geyer and co-workers [3].

This is indeed the case of the state (18). Let us now discuss the structure of the spurious states which are introduced by the use of the coherent state. Following the arguments of [3], it can be shown that none is present if a level is less than half-filled and that spurious states do appear when a level is

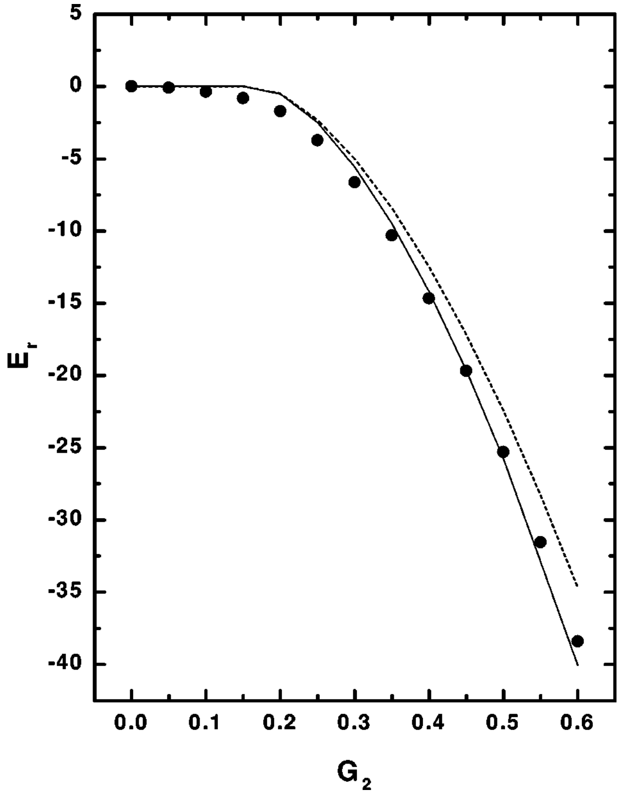

FIG. 2. Real part of the potential energy surface, E_r, corresponding to the case $G_1=0$, $G_2=G_3$. The parameters of the model are given in the text. With solid lines we represent the results of the calculations using coherent states, dashed lines represent the results of the mean field approach, and the dots correspond to the results of an exact diagonalization. Both quantities, E_r and G_2, are given in units of GeV.

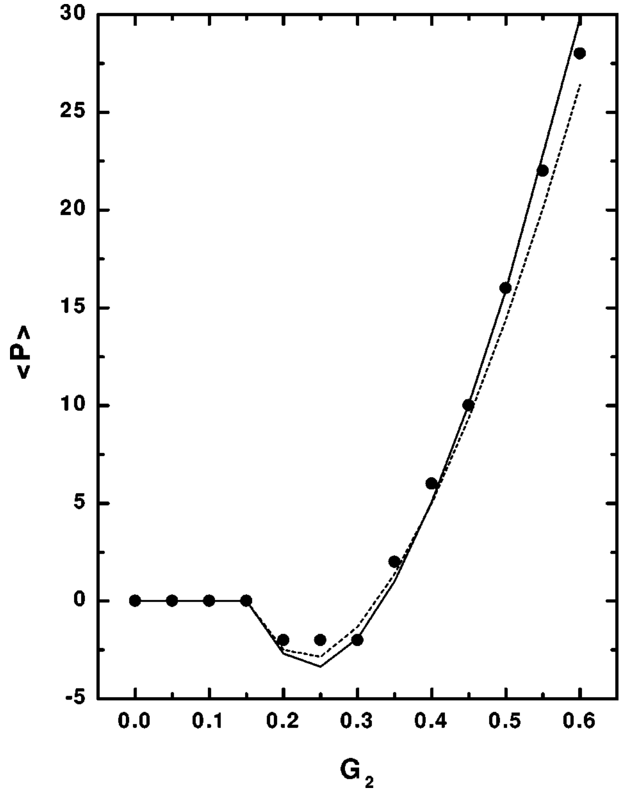

FIG. 3. Expectation value of the symmetry operator, $\langle P\rangle$, as a function of G_2 for the case $G_1=0$, $G_2=G_3$. The parameters of the model are given in the text and G_2 and G_3 are given in units of GeV. The results are represented with the notation given in the caption to Fig. 2

more than half-filled. The dominant part still comes from the Pauli-allowed states. That is the reason why, up to $N=2\Omega$, we do not expect to find large differences with respect to the exact solution (as we shall show explicitly later on; see the discussion which accompanies the results of Figs 2 and 3).

In order to identify spurious states, we shall discuss the group chain involved in the calculations. It is

$$U(4\Omega)\supset U(2\Omega)\times U_T(2), \tag{19}$$

where 2Ω is the degeneracy of each level and $U_T(2)$ distinguishes between the upper and lower levels. In order to have a complete antisymmetric state, the irreducible representation (irrep) of $U(4\Omega)$ has to be antisymmetric, i.e., the Young diagrams of the two groups to the right in (19) have to be conjugate to each other. When we denote by $[h_1h_2]$ the irrep of $U_T(2)$, the one of $U(2\Omega)$ should be $[2^{h_2}1^{(h_1-h_2)}]$. The generalized quasispin label, T, can be deduced from there and its value is given by $T=(h_1-h_2)/2$. As a special case let us consider the case of two particles $(2p)$, as created by the generators of SO(5). The L_+ creates a $2p$ state with $T=1$. Applying the operator S_- to it (once and twice) one obtains the $T_z=0$ and $T_z=-1$ components of $T=1$, corresponding to a $1p-1h$ and a $2h$ state, respectively. In terms of a Young diagram this corresponds to $[2]$ for $U_T(2)$. Unphysical states of the boson mapping occur when the Young diagram of $U_T(2)$ has more than 2Ω columns, because the conjugate Young diagram for $U(2\Omega)$ has more than 2Ω rows and it is not allowed. In consequence, no antisymmetric state in the fermion space corresponds to this case. The relation of h_1 and h_2 to the total number of particles N is given by $N=h_1+h_2$ and the one for the generalized quasispin T is given above. The restriction in the number of rows, to be less than or equal to 2Ω rows, in the Young diagram of $U_T(2)$, i.e., $h_1\leqslant 2\Omega$, gives

$$T\leqslant \min\left\{2\Omega-\frac{N}{2},\Omega\right\}. \tag{20}$$

Next we discuss a simple example, given in Ref. [3]. There the $4p-4h$ case for $\Omega=2$ and 3 is discussed. $4p-4h$ corresponds to $N=8$. The possible allowed Young diagrams [taking into account that in $U_T(2)$ only diagrams of the form $[2n_1,2n_2,2n_3]$ are allowed because the elemental one is $[2]$; see Ref. [4]], are

$$[8]\ \ (T=4),\ \ [6,2]\ \ (T=2),\ \ [4^2]\ \ (T=0) \tag{21}$$

all for $U_T(2)$. For $\Omega=2$ the restriction (20) corresponds to ($N=8$ and $\Omega=2$) $T=0$ only. Thus the other states in (21) with $T>0$ correspond to nonphysical states. For $\Omega=3$ the condition (20) reads $T \leqslant 2$ and thus only $T=0$ and 2 are physical states. The one with $T=4$ is unphysical. For $\Omega \geqslant 4$ all states are physical.

The maximal value of T which can be formed is Ω. When the number N of particles and holes is smaller than or equal to 2Ω, all states are physical. The coherent state that we have used in the present case is not coupled to a definite value of T. One can think of a coupling scheme where a definite value of T is kept and all other values are excluded, i.e., by enforcing the condition (20) which would take into account the Pauli principle completely. However, this would imply a highly involved algebra.

We preferred to perform a variational treatment with the already defined coherent state and we check the procedure afterward.

D. Explicit evaluation of the expectation values

Using the above defined coherent state we can easily calculate the expectation value of the interactions appearing in the Hamiltonian. The expectation value of an arbitrary product of creation and annihilation operators of the bosonic part is given by

$$\langle \alpha | (B_i^\dagger)^{n_i} (B_i)^{m_i} | \alpha \rangle = (\alpha_i)^{n_i+m_i} e^{-i\varphi_i(n_i-m_i)}, \tag{22}$$

where $i=b,p,h$.

The result for the fermionic part is

$$\begin{aligned}\frac{\langle \chi | (b_F^\dagger)^{n_{F_1}} (b_P^\dagger)^{n_{P_1}} (b_H^\dagger)^{n_{H_1}} (b_F)^{n_{F_2}} (b_P)^{n_{P_2}} (b_H)^{n_{H_2}} | \chi \rangle}{\langle \chi | \chi \rangle} &= (\chi_F)^{n_{F_1}+n_{F_2}} e^{-i\xi_F(n_{F_1}-n_{F_2})} (\chi_P)^{n_{P_1}+n_{P_2}} \\ &\quad \times e^{-i\xi_P(n_{P_1}-n_{P_2})} (\chi_H)^{n_{H_1}+n_{H_2}} e^{-i\xi_H(n_{H_1}-n_{H_2})} \\ &\quad \times \Omega^{n_{F_2}+n_{P_2}+n_{H_2}} (F_{n_{F_2} n_{P_2} n_{H_2}} / F_{000}),\end{aligned} \tag{23}$$

with

$$F_{n_{F_2} n_{P_2} n_{H_2}} = \sum_{n_F=0}^{2\Omega - n_{F_2}} \sum_{n_P=0}^{\Omega - [(n_F+n_{F_2}+1)/2] - n_{P_2}} \sum_{n_H=0}^{\Omega - [(n_F+n_{F_2}+1)/2] - n_{H_2}} \frac{(\chi_F)^{2n_F}}{n_F!} \frac{(\chi_P)^{2n_P}}{n_P!} \frac{(\chi_H)^{2n_H}}{n_H!} 2^{n_F} \Omega^{n_F+n_P+n_H}. \tag{24}$$

After applying this procedure to the Hamiltonian of Eq. (1), the real part of the PES is written

$$E_r = \Omega(F_1+F_2+F_3+F_4+F_5+F_6+F_7+F_8+F_9+F_{10}), \tag{25}$$

with

$$F_1 = \omega_F(\chi_P^2 F_{010} + \chi_H^2 F_{001} + 2\chi_F^2 F_{100})/F_{000},$$

$$F_2 = \omega_b \alpha_b^2 + \omega_p \alpha_p^2 + \omega_h \alpha_h^2,$$

$$\begin{aligned}F_3 = G_1 \alpha_b [&\cos(-\phi_b - \xi_F + \xi_H) 2\chi_F \chi_H (F_{101}/F_{000}) \\ &+ \cos(-\phi_b - \xi_P + \xi_F) 2\chi_P \chi_F (F_{110}/F_{000})],\end{aligned}$$

$$\begin{aligned}F_4 = G_1 \alpha_b [&\cos(\phi_b - \xi_F + \xi_P) 2\chi_F \chi_P (F_{110}/F_{000}) \\ &+ \cos(\phi_b - \xi_H + \xi_F) 2\chi_H \chi_F (F_{101}/F_{000})],\end{aligned}$$

$$\begin{aligned}F_5 = G_2 \alpha_h \cos(-\phi_h - \xi_P) \chi_P [&(F_{010}/F_{000}) - \chi_P^2 (F_{020}/F_{000}) \\ &- 2\chi_F^2 (F_{110}/F_{000})],\end{aligned}$$

$$F_6 = -G_2 \alpha_h \cos(-\phi_h - 2\xi_F + \xi_H) \chi_F^2 \chi_H (F_{201}/F_{000}),$$

$$F_7 = G_2 \alpha_h \cos(\phi_h + \xi_P) \chi_P (F_{010}/F_{000}),$$

$$\begin{aligned}F_8 = G_3 \alpha_p \cos(-\phi_p - \xi_H) \chi_H [&(F_{001}/F_{000}) - \chi_H^2 (F_{002}/F_{000}) \\ &- 2\chi_F^2 (F_{101}/F_{000})],\end{aligned}$$

$$F_9 = -G_3 \alpha_p \cos(-\phi_p - 2\xi_F + \xi_P) \chi_F^2 \chi_P (F_{210}/F_{000}),$$

$$F_{10} = G_3 \alpha_p \cos(\phi_p + \xi_H) \chi_H (F_{001}/F_{000}). \tag{26}$$

The imaginary part of the PES is given by a similar expression where in the definitions of $F_3 \rightarrow F_{10}$ the factors cos( ) are replaced by sin( ). As was done for the case of the model of Schütte and Da Providencia (see Sec. II A) and without loss of generality, we can fix all angles at zero and determine the amplitudes of the order parameters from the minimization of the PES.

III. RESULTS

The minimization of the PES, for the general case with all G_i different from zero, has been performed numerically by using a routine of the CERN library [28]. All couplings between bi-fermions and bosons are included and for the sake of convenience the rescaled coupling constants x_1, x_2, and x_3 are introduced:

$$x_1 = G_1 \sqrt{\frac{2\Omega}{\omega_F \omega_b}}, \quad x_2 = G_2 \sqrt{\frac{\Omega}{\omega_F \omega_h}}, \quad x_3 = G_3 \sqrt{\frac{\Omega}{\omega_F \omega_p}}. \tag{27}$$

We have found different types of condensates for the following cases:

(i) $x_1 > 1$, x_2 and x_3 arbitrary and $\omega_F = \omega_p = \omega_h = \omega_b = 1$ GeV.

(ii) $x_1 > 1$, x_2 and $x_3 < 3$ and $\omega_F = \omega_p = \omega_h = 1$ GeV, $\omega_b = 10$ GeV.

(iii) $x_1>1$, $x_2>3$, $x_3>3$ and $\omega_F=\omega_p=\omega_h=1$ GeV, $\omega_b=10$ GeV.

(iv) $x_1>1$, $x_2>3$, $x_3>3$ and $\omega_F=\omega_b=1$ GeV, $\omega_p=\omega_h=10$ GeV.

As it is discussed below, the just introduced energy scale is taken from [29]. All these cases correspond to $\Omega=15$ ($N=30$). In case (i) the minimization leads to a boson condensate dominated by b bosons without significant contributions from the pp- and hh-pair sectors of the Hamiltonian. In case (ii) the minimization shows the existence of a fermion condensate dominated by ph pairs. The solution which yields a condensate of b bosons can be excluded by choosing a large value of ω_b and small coupling constants $x_2=x_3$. The new solutions, which are consistent with the set of parameters (iii) and (iv), are dominated by the pp and hh channels. As it turns out, the pp and hh channels contribute equally. This is a consequence of the fact that $x_2=x_3$ for all cases. In general this is not true. However, considering quarks and antiquarks, represented as particles and holes, respectively, these two coupling constants should be equal due to charge conjugation. In other applications, however, the coupling constants can be chosen in a different way. Due to the choice of $x_2=x_3$, the expectation value of the symmetry operator R,

$$R=B_h^\dagger B_h-B_p^\dagger B_p-\frac{1}{2}(\nu-\overline{\nu}),$$

vanishes at the minimum. We have also considered, in detail, the appearance of condensates for the case

$$\omega_F=0.4,\quad \omega_p=\omega_h=1,\quad G_1=0,\quad G_2=G_3=0\rightarrow 0.5 \tag{28}$$

(all quantities expressed in GeV), which corresponds to quarks and antiquarks at an energy of 0.4 GeV (i.e., given the energy in units of GeV) and p and h bosons at an energy of 1 GeV. These values correspond to the assumption that the first glueball, consisting of two gluons, has an energy of about 2 GeV and that the three quark system has an energy of about 1.2 GeV. These parameters have been fixed, following the findings reported in Ref. [29], where the structure of glueballs, as described in lattice gauge calculations, was reproduced using an effective simple model.

This is clearly a very rough assumption because pure glueballs may not exist but they will mix with quarks and antiquarks and their energy will not exactly be 2 GeV. The choice of ω_F can be understood as a mean value of the lowest nuclear resonances. Moreover, this toy model does not have the same number of degrees of freedom of QCD. Therefore, a direct comparison to QCD should be done with caution, though the main structures are present. In our example the coupling between b bosons and ph bi-fermions is excluded because it always leads to a global minimum. Under these conditions the interaction between p and h boson and pp and hh bi-fermions yields a local minimum. One might interpret it as produced by a ground state dominated by a ph condensate. At higher energies a phase transition to a pp and hh condensate develops.

Figure 2 shows the real part of the PES as a function of the interaction strength G_2. Figure 3 shows the expectation value of the symmetry operator P,

$$P=B_b^\dagger B_b+B_p^\dagger B_p+B_h^\dagger B_h-\frac{1}{2}(\nu+\overline{\nu}),$$

as a function of the interaction strength G_2. Positive values of $\langle P\rangle$ indicate a boson-dominated ground state and negative values of $\langle P\rangle$ indicate the fermion character of the ground-state correlations. As it is shown in Fig. 3, for values of $G_2<0.15$ GeV the expectation value of the symmetry operator is $\langle P\rangle=0$ and it means that the local ground state is the uncorrelated ground state. For $0.15\ \mathrm{GeV}<G_2<0.33$ GeV, the ground state is dominated by pairs of fermions while for $G_2>0.33$ GeV the vacuum is a boson condensate.

In order to illustrate the validity of the approximations, we have calculated the potential energy surface (PES) and the eigenvalue of the P symmetry in (i) a mean field approach and (ii) an exact form, consisting of the diagonalization of H in a certain subspace. The results corresponding to the mean field approach have been obtained by following the method outlined in Ref. [18]. In this reference, the fermionic fields are transformed by using Hartree-Bogoliubov transformations and the bosons are shifted in order to include nonzero vacuum expectation values. The resulting Hamiltonian is solved by using the equation of motion method. Concerning the exact solution, we have performed a Lanczos diagonalization in a restricted subspace corresponding to the zero eigenvalue of the R symmetry, which for the case of $G_2=G_3$ exhausts the relevant components of the wave function. For any other point in the parametric space, i.e., $G_2\neq G_3$, one should include all eigenvalues of the symmetry.

The results are shown in Figs. 2 and 3. From these results one can conclude that the comparison between exact results and the variational ones is rather good, a fact which should also indicate the order of magnitude of the effects induced by spurious states. Obviously, these results cannot be used to support the claim that spurious contributions are removed from the calculations. Rather, as it is usually the case of the variational approach, the spurious components of the wave functions will certainly affect the structure of the spectrum but not the ground-state energy, as we have shown. This is also the case for the conventional many-body treatment, say, of pairing correlations in the BCS theory, as well as in the case of a model which belongs to a lower group symmetry [24].

Figure 4 shows the amplitude of the bosonic order parameter α_p and Fig. 5 shows the amplitude of the fermionic order parameter χ_P, both as a function of the interaction strength G_2. The phase transition point around the value $G_2=0.2$ GeV is clearly seen. The order parameter χ_P is positive, and it saturates, while the order parameter α_p decreases steadily with increasing values of G_2. These results show that for an intermediate range of values of the interaction strength G_2, a fermion condensate is formed. For larger values of G_2 the boson condensate always dominates. As was said before, the expectation value of the symmetry R vanishes because of $G_2=G_3$.

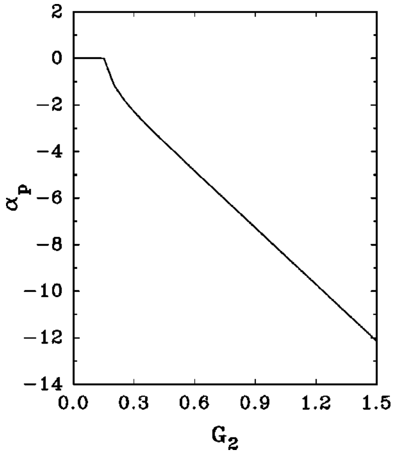

FIG. 4. Amplitude of the bosonic order parameter $\alpha_p = \alpha_h$, as a function of the coupling constant $G_2 = G_3$, for the case $G_1 = 0$ and for the model parameters discussed in the text. The coupling constant G_2 is given in units of GeV.

Finally we have investigated the dependence of the gap

$$\Delta = \langle L_+ \rangle, \tag{29}$$

which is the expectation value of the pp bi-fermion operator, as a function of the order parameter. The results are shown in Fig. 6. For small values of χ_P the gap increases very fast, it reaches a maximum at approximately $\chi_P = 0.5$, and afterwards it decreases rapidly. For large values of χ_P a saturation is obtained. This feature is an artifact of the procedure, since for large values of G_2 the number of fermion pairs does not change. It is interesting to note that, as the result of the minimization, we have always obtained values of χ_P around the value corresponding to the maximum of Δ. This result, independent of the value of the order parameter α_p, indicates that the system maximizes the gap.

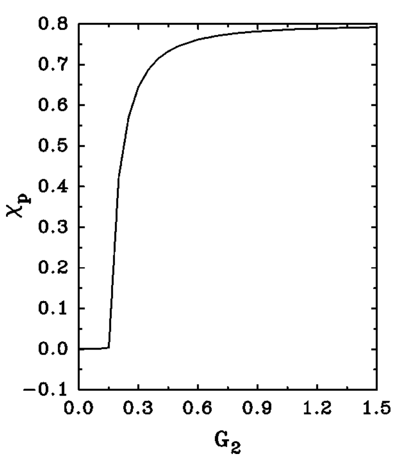

FIG. 5. Amplitude of the fermionic order parameter $\chi_P = \chi_H$, as a function of the coupling constant $G_2 = G_3$, given in units of GeV, for the case $G_1 = 0$ and for the model parameters discussed in the text.

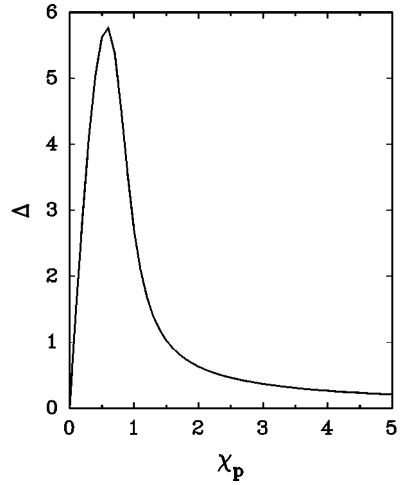

FIG. 6. Gap parameter $\Delta = \langle L_+ \rangle$, of Eq. (29), as a function of the fermionic order parameter χ_P.

IV. CONCLUSIONS

In this work we have investigated the appearance of fermion and boson condensates in a system described by pairs of fermions coupled to bosons. The model is a generalization of the bi-fermion Hamiltonian due to Geyer [1] and of the fermion-boson coupling of Schütte and Da Providencia [7]. The model has exact solutions which correspond to the $Sp(4,R)$ symmetry representation. We have performed an exact Dyson boson mapping of the Hamiltonian. This mapping procedure preserves the algebra of the generators of the group. Trial states were expressed in terms of coherent states with complex order parameters. The Pauli principle was approximately obeyed in constructing boson images of the trial state. Matrix elements of the transformed Hamiltonian acting on trial states have been used to calculate potential energy surfaces. The minimization of the potential energy surface leads to different values of the order parameters, as functions of the coupling strength of the model, which have been interpreted as the signatures of phase transitions. We have shown that the model, which reproduces correctly the results of Schütte and Da Providencia as a limiting case, predicts the appearance of fermion and boson condensates.

For the case in which realistic masses for the quarks and for the gluons were used, we obtained a fermion condensate for only a small range of the interaction parameter G_2. Except for very small interaction strength, the boson condensate always dominates. This does not mean that only bosons are present in the condensate but rather a mixture of ph fermions with bosons, such that the expectation value of the symmetry operator P, which is the difference of boson number to the particle and hole fermions, is positive. However, a large value of $\langle P \rangle$ indicates an overwhelming dominance of the bosons. The characteristics of the phase transitions of the cases studied above and depicted in Fig. 3 depend mainly on the interaction strength and not so much on the relative energies. The result of Fig. 3 was obtained by excluding the boson condensate of type b, which means that we investigated a local minimum at a higher energy than the ground-state minimum. This might also indicate a phase transition of

QCD at higher energies. No detailed conclusions can be drawn because of the schematic character of our model.

Due to the nonperturbative character of the method which we have used to find the condensates, it may be relevant in the analysis of QCD at low energies. The model contains all basic ingredients, namely the interaction between particle (quarks) and hole (antiquarks) fermions with bosons (gluons) with different degrees of freedom (color and flavor). Of course, the version used here can only be used as a toy model to understand phase transitions in QCD. For a more realistic treatment the correct number of degrees of freedom has to be used, instead, without changing the procedure. The above presented case has the advantage that the solutions are obtained in an algebraic way.

The Hamiltonian used in this work may be relevant in an area completely remote to the QCD, namely high-temperature superconductivity [10]. There the Hamiltonian is more directly related, in form and structure, to the one used in the present work.

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