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On the mean value of the energy for resonant states

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Abstract

In this work we discuss possible definitions of the mean value of the energy for a resonant (Gamow) state. The mathematical and physical aspects of the formalism are reviewed. The concept of rigged Hilbert space is used as a supportive tool in dealing with Gamow-resonances.

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1. Introduction

The use of resonant (Gamow) states in nuclear structure calculations was proposed years ago by the Stockholm–Debrecen group [1,2] and since then the notion has been widely applied to a variety of physical situations, with a remarkable success. The reader is kindly referred to Refs. [1,2] and references therein for a comprehensive introduction on the subject. Although Gamow's idea of decaying states was immediately recognized as a major breakthrough in Quantum Mechanics [3], its use in modern nuclear structure calculations was delayed for nearly forty years until the work of Tore Berggren [4] shows that single-particle basis (Beggren's basis) can accommodate single particle resonant states of complex energy. Berggren's suggestions were adopted by Liotta and co-workers [5] thus given structural identity to a modern view of the continuum and its effects upon nuclear structure observables [2]. Mathematically speaking the Stockholm approach is, as a matter of fact, based on the identification of single-particle resonances, in the standard one-body nuclear central potential [6], and in the numerical calculation of the needed matrix elements of nuclear two-body interactions [5]. So far, the numerical treatment of resonant states, either by performing explicitly the needed integrals in the complex plane [4,6] or by projecting them on the real axis as non-overlapping states with a Breit–Wigner broadening, has proved its feasibility. A parallel formal development on Gamow's resonances followed from the work of Bohm and Gadella, which is docu-

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mented in a rigorous mathematical way in [7]. However, a link between these two fronts of research on Gamow resonant states is missed. Particularly, we have noticed that basic notions, as the one of the value of the expectation value of the Hamiltonian on a resonant state, still need to be clarified and/or discuss in detail. In this paper we focus on the question about the definition of the expectation value of the Hamiltonian on a resonant state. We shall illustrate this point as follows. In ordinary Quantum Mechanics the mean value of the energy on a state represented by the density ρ is given by $\mathrm{Tr}\,\rho\mathrm{H}$, where H is the Hamiltonian. If ρ represents a pure state it has the form $\rho = |\psi\rangle\langle\psi|$ and the quantity $\mathrm{Tr}\,\rho\mathrm{H}$ gives the usual formula for the mean value of the energy on the state $|\psi\rangle$, which is written as $\langle\psi|H|\psi\rangle$. Gamow resonant states (or Gamow states, for short), are state vectors representing exponentially decaying states. They are not ordinary quantum states. Such a quantum state has a Breit–Wigner energy distribution, which is strictly nonzero for all values of the energy E, i.e. $-\infty < E < \infty$. This contradicts the fact that the spectrum of the physical Hamiltonian H should be lower-bound. Therefore, the Gamow state cannot be represented as a vector in a Hilbert space in which H is a self-adjoint operator [9–12]. In addition, if φ is a Gamow state, it must fulfill a condition of the kind $e^{-itH}\varphi = Ae^{-\gamma t}\varphi$, A being a phase ($A = e^{\phi(t)}$ with $\phi(t)$ real). Thus, $H\varphi = (\phi(t)/t + i\gamma)\varphi$ and the Gamow vectors must be eigenvectors of the total Hamiltonian with complex eigenvalues. Obviously this condition cannot be satisfy in the ordinary Hilbert space, since H must be self-adjoint. In fact, Gamow states can be properly defined as functionals on a certain space of test vectors, in the same form that the generalized eigenstates of the position and momentum operators are defined [11,12]. The space of functionals contains also the Hilbert space of ordinary states, so that no information is lost from ordinary Quantum Mechanics. Thus, we have a triplet of spaces $\Phi \subset \mathscr{H} \subset \Phi^{\times}$, called the rigged Hilbert space or Gelfand triplet.

Here $\mathscr{H}$ is the Hilbert space, Φ the space of test vectors and $\Phi^{\times}$ its antidual[1]. In this context, Gamow states will exist in the dual space $\Phi^{\times}$ [7,13]. At this point a difficulty arises, namely: $\Phi^{\times}$ is not an inner product space. If we represent the Gamow vector as $|f_0\rangle$, the bracket $\langle f_0|H|f_0\rangle$ and the bracket $\langle f_0|f_0\rangle$ are not defined. Thus, in principle, we cannot define the mean value of H on $|f_0\rangle$ as we would do in ordinary Quantum Mechanics. This is precisely the sort of questions which we meant above.

The aim of the present paper is to recall on the attempts to define the mean value of H on a Gamow state and discuss their advantages or disadvantages. These attempts use the following concepts:

1. The mean value of the energy of a decaying state must be zero, because the energy of a decaying process should be invariant. A nonzero energy will be in contradiction with the principle of conservation of the energy [14–16].
2. The energy average of a Gamow state should be complex because the Gamow state is an eigenvalue of H with complex energy.

[1] $F \in \Phi^{\times}$ if it is a mapping from Φ into the complex plane C with the following conditions: (i) Antilinearity: $F(\alpha\varphi + \beta\phi) = \alpha^* F(\varphi) + \beta^* F(\phi)$, for all $\alpha, \beta \in C$ and all $\varphi, \phi \in \Phi$. (ii) Continuity: F must be a continuous mapping. $\Phi^{\times}$ is a topological vector space with a topology which is coarser than the Hilbert space topology on $\mathscr{H}$.

3. If we admit that Gamow states are genuine quantum states they must have a real energy average that should be determined from first principles [4,11,17].
4. Gamow states admit a representation as **normalizable** vectors in a Hilbert space in which the Hamiltonian is **not** a self-adjoint operator and it has an spectrum of eigenvalues extending from $-\infty$ to ∞. In this case, we can define scalar products of Gamow vectors and a mean value of the energy, which is real [11].

In the last section, we shall present and evaluate these four possibilities. Before, in the next section, we recall the definition of a Gamow vector and enumerate some of its properties.

2. The Gamow vectors and their properties

In ordinary Quantum Mechanics, it is customary to associate a normalizable vector (with unitary norm) to each pure quantum state. If we have a quantum decaying state and we want to assign to it a vector ψ in a separable Hilbert space we can define its *nondecay probability* for $t > 0$ as [11,12,18]

$$P(t) = |A(t)|^2 , \quad \text{where } A(t) = \langle \psi | e^{-itH} | \psi \rangle . \tag{1}$$

Since ψ is normalized, one has $0 \leq P(t) \leq 1$ and the decay probability is given by $1 - P(t)$. $A(t)$ is called the nondecay amplitude. It has been shown that $A(t)$ is roughly exponential for values of time which are not too short neither too long. At $t = 0$ the time derivative of $A(t)$ is zero and this excludes the exponential time behaviour of ψ for small times. For large values of time $A(t)$ goes to zero slower than an exponential [18].

This is what the theory predicts *provided that a decaying state can be represented by a vector in a Hilbert space*. We have already mentioned in the introduction another argument to show that an exponentially decaying state cannot be represented by a vector in a Hilbert space.

The experimental evidence on decaying states shows a decay which seems to be exponential up to the degree of experimental accuracy. In consequence, it is natural to consider exponentially decaying vector states as true physical states. But, we have to find out the mathematical nature of these objects. This problem has been satisfactorily solved by Arno Bohm and coworkers, in terms of rigged Hilbert spaces or Gelfand triplets [7,19–24].

In fact, we can look at a decaying process as the second half of a resonant scattering process in which the process of formation of a resonance is ignored [11]. Resonances can be defined in several ways, but it is generally accepted that, under rather general conditions, we can associate a resonance to each of the pairs of poles of the analytic continuation of the S-matrix, $S(E)$, on the real semi-axis of the energy values [9–11,18,25]. In many implementable physical cases we can assume that in the resonant scattering process both the ''free'' Hamiltonian H_0 and the ''perturbed'' Hamiltonian $H = H_0 + V$, where V is a suitable potential, have the same continuous spectrum. This spectrum coincides with the positive semi-axis $R^+ = [0,\infty)$ and the identity of both continuous spectra is a consequence of the asymptotic completeness of the scattering.

This property implies that the Møller operators $\Omega_{\pm}$ exist and fulfill the following relation [7]:

$$\Omega_{\pm} H_0 = H \Omega_{\pm} \quad \text{or} \quad H = \Omega_{\pm} H_0 \Omega_{\pm}^{\dagger} . \tag{2}$$

Furthermore, to simplify the formalism, we can assume that the energy spectrum is nondegenerate (which is the case of a spherically symmetric potential in the $l = 0$ channel). In this case, it exists a unitary operator which connects the abstract Hilbert space of states and $L^2(R^+)$ (the Hilbert space of the energy representation) such that $UH_0U^{-1}\phi(E) = E\phi(E)$, where U *diagonalizes* the free Hamiltonian H_0.

The explicit construction of the rigged Hilbert spaces for the decay process is relevant in our discussion and, therefore, we want to summarize it here. Further details can be found in the literature [7,13,21–23]. First, we take the spaces of the so-called *very well behaved functions*. These spaces are defined by analytic functions on the upper or the lower half planes of the complex plane C that vanish at infinity. The boundary values of these functions on the positive semi-axis R^+ are uniquely defined and, furthermore, it is possible to recover all the values of these functions knowing their boundary values on R^+ [26]. Let us call $\Delta_{\pm}$ the spaces of these boundary values. We can construct a rigged Hilbert space suitable for the definition of decaying Gamow vector as follows. Firstly, we define $\Phi_+ = \Omega_+ U^{-1} \Delta_+$, where Ω_+ is the outgoing Møller operator. Then, if $\mathscr{H}$ is the Hilbert space of scattering states[2], we have the following rigged Hilbert space:

$$\Phi_+ \subset \mathscr{H} \subset (\Phi_+)^{\times} . \tag{3}$$

Since every function in Δ_+ determines *uniquely* an analytic function on the upper half plane for every vector $\varphi_+ \in \Phi_+$ we have an analytic function $\phi(E)_+$ on the upper half plane. Its complex conjugate $\phi_+^{\#}(E) = [\phi_+(E)]^*$ is an analytic function on the *lower half plane*. Then, if $z_R = E_R - i\Gamma/2$ denotes a resonance pole $(\Gamma > 0)$ the Gamow vector $|f_0\rangle$ can be defined as a functional on Φ_+ as[3]

$$\langle \varphi_+ | f_0 \rangle = \phi_+^{\#}(z_R) . \tag{4}$$

This definition implies several important properties namely: (i) $|f_0\rangle \in \Phi_+^{\times}$ is a continuous anti-linear functional on Φ_+, (ii) $|f_0\rangle \notin \mathscr{H}$, (iii) the Hamiltonian H satisfies $H\Phi_+ \subset \Phi_+$ and is continuous with the topology on Φ_+. It means that H can be continuously extended (with the weak topology) to a continuous operator on $\Phi_+^{\times}$, so that the action of H on $|f_0\rangle$ is a well-defined operation. Then, we have $H|f_0\rangle = z_R|f_0\rangle$, which means that the Gamow vector $|f_0\rangle$ is an eigenvector of H with eigenvalue z_R. This is possible because $\Phi_+^{\times}$ is not a Hilbert space, and, (iv) the time evolution of $|f_0\rangle$ is only possible for positive values of t. We can show that [7,13]

$$e^{-itH}|f_0\rangle = e^{-itE_R} e^{-\Gamma t}|f_0\rangle . \tag{5}$$

This means that $|f_0\rangle$ decays exponentially. All these properties justify the choice of $|f_0\rangle$ as a representation of the decaying Gamow vector.

In a resonant scattering process, together with the decaying channel, it exists a process of *capture* or formation of a resonance. This is called the growing or capture

[2] I.e., the absolutely continuous space for the total Hamiltonian H.

[3] Here, we are using the notation in [27].

process and it can be described by the evolution of another functional, which is the growing Gamow vector $|\tilde{f}_0\rangle$[4]. This is a functional on Φ_- and therefore an element of the vector space $\Phi_-^\times$. As functional its definition coincides with the one given in Eq. (4) by replacing the function $\phi_+(E)$ by $\phi_-(E)$, which is analytic on the lower half plane, and the point z_R by its complex conjugate $z_R^* = E_R + i\Gamma/2$,

$$\langle \varphi_- | \tilde{f}_0 \rangle = \phi^\#(z_R^*) \,. \tag{4'}$$

Its properties are the following: (i) $|\tilde{f}_0\rangle \notin \mathscr{H}$, (ii) $H|\tilde{f}_0\rangle = z_R^* |\tilde{f}_0\rangle$ and, (iii) its time evolution is well defined if $t < 0$,

$$e^{-itH}|\tilde{f}_0\rangle = e^{-itE_R}\, e^{\Gamma t}|\tilde{f}_0\rangle\,, \tag{6}$$

i.e., $|\tilde{f}_0\rangle$ increases exponentially until $t = 0$, which is conventionally the time at which the capture process is completed and the decay starts.

From these definitions it can be concluded that Gamow vectors obey

$$\langle f_0 | \varphi_+ \rangle = \langle \varphi_+ | f_0 \rangle^* \quad \text{and} \quad \langle \tilde{f}_0 | \varphi_- \rangle = \langle \varphi_- | \tilde{f}_0 \rangle^*\,, \tag{7}$$

where the star denotes complex conjugation. The bra vectors $\langle f_0|$ and $\langle \tilde{f}_0|$ are continuous *linear* functionals on Φ_+ and Φ_-, respectively,

$$\langle f_0 | H = z_R^* \langle f_0 |, \qquad \langle \tilde{f}_0 | H = z_R \langle \tilde{f}_0 |. \tag{8}$$

The time evolution of $\langle f_0|$ is defined for $t > 0$ only and it gives

$$\langle f_0 | e^{itH} = \langle f_0 | e^{itE_R}\, e^{-\Gamma t} \tag{9}$$

and the time evolution of $\langle \tilde{f}_0|$ is defined for $t < 0$ as

$$\langle \tilde{f}_0 | e^{itH} = \langle \tilde{f}_0 | e^{itE_R}\, e^{\Gamma t}. \tag{10}$$

In this context, it seems that exponentially behaving state vectors are the only class of vectors which can represent Gamow states. To construct a representation, one has to consider these states and also a *background*, physically produced by the interaction with the environment, re-scattering processes, etc. [18], and mathematically by contour integrals in the complex plane [11,19–23]. Usually, the resonant scattering process is produced by the interaction of a free prepared state with a potential, creating the resonance. The prepared state must be represented by a Hilbert space vector, say φ. Then, if $\Omega_- \varphi = \varphi_-$, we can show that φ_- is the sum of two contributions. One of these contributions is proportional to the decaying Gamow vector $|f_0\rangle$ and the other to the background, represented by a vector in $\Phi_+^\times$, so that [11,19–23]

$$\varphi_- = a|f_0\rangle + |\text{background}\rangle\,, \tag{11}$$

where a is a complex number. This background would be responsible for the deviations of the exponential law on the range of short and large times.

Our next goal is to present and compare the definitions of the energy average on Gamow vectors.

[4] The question about the physical meaning of this functional, i.e: whether it is related to the capture or creation of a resonance, has not been answered yet. If capture and decaying processes are not equally probable they can not be symmetric in the sense presented here. Thus, in this interpretation, $|\tilde{f}_0\rangle$ would be the time reversal of the Gamow vector $|f_0\rangle$.

3. Definitions of energy average on Gamow vectors

In this section we shall review the known results obtained in dealing with the definition of energy averages on Gamow states, which are available in the literature, as well as our own definition of it.

3.1. The mean value of the energy is equal to zero

It was Nakanishi who first proposed this idea [14]. In fact, if $H|f_0\rangle = z_R|f_0\rangle$ and $\langle f_0|H = z_R^*\langle f_0|$, we have that $\langle f_0|H|f_0\rangle = z_R\langle f_0|f_0\rangle = z_R^*\langle f_0|f_0\rangle$. This implies that $(z_R - z_R^*)\langle f_0|f_0\rangle = 0 \Rightarrow \langle f_0|f_0\rangle = 0$ and, therefore, $\langle f_0|H|f_0\rangle = 0$. The weak point of this argument is that the bracket $\langle f_0|f_0\rangle$ is not defined. Further attempts to define it have been made [28], but the results are not convincing from a mathematical point of view.

3.2. The averages are complex

In specific models, like Friedrichs's model, the bracket $\langle \tilde{f}_0|f_0\rangle$ is well defined and its value is one [29]. If we try to obtain this result in a general model independent setting we conclude that $\langle \tilde{f}_0|f_0\rangle$ can be defined as a distribution-kernel and that it has the value one, although it is not clear if this is the unique choice [30]. If we now define $\Pi = |f_0\rangle\langle \tilde{f}_0|$, it is now obvious that $\Pi^2 = \Pi$. This idempotency suggest us that that Π could be taken as the density operator for the decaying Gamow vector $|f_0\rangle$. Now if it would be possible to define $\mathrm{Tr}\{H\Pi\}$ and this would be a candidate for the average value of H on $|f_0\rangle$. In fact, with the help of some generalized spectral decompositions [30] for the Hamiltonian in terms of the Gamow vectors and the generalized eigenvectors of H with eigenvalues in the continuous spectrum of H, we can define this trace in such a way that $\mathrm{Tr}\{H\Pi\} = \langle \tilde{f}_0|H|f_0\rangle$ [30], thus

$$\langle \tilde{f}_0|H|f_0\rangle = z_R\langle \tilde{f}_0|f_0\rangle = z_R\,. \tag{12}$$

Yet this result cannot be acceptable from the physical point of view. Due to the time-energy uncertainty principle we cannot measure *simultaneously* the real part of z_R, which is the resonant energy, and its imaginary part, which is proportional to the inverse of the half life. Thus, z_R cannot be the *average* of any measurement process and cannot be accepted as the energy average. Also, from these considerations, we conclude that the energy average of a Gamow vector, if it can be defined, should be real.

3.3. The averages are real in the interpretation of Bohm

This point of view is based in the idea that it is possible to construct a rigged Hilbert space, in which the Gamow vector is a vector in the Hilbert space, under the following conditions:

1. The continuous spectrum of H is the whole real axis,
2. H is not self-adjoint (although it is still symmetric, i.e., $\langle \phi|H\psi\rangle = \langle H\phi|\psi\rangle$ for all ϕ and ψ in the domain of H), and,

3. from the point of view of the Hilbert space, the Gamow vector is not in the domain of H, but the action of H on the Gamow vector is well defined in the dual space (that includes the Hilbert space).

The spaces of analytic functions on a half plane that we are using here are spaces of Hardy functions [31–34]. These functions are determined by their boundary values on the positive semi-axis $R^+ = [0,\infty)$. Moreover, we have an explicit formula to recover their values on the half plane (including the negative semi-axis, $R^- = (-\infty,0]$) from their values on the positive semi-axis [26]. If we denote by $\mathscr{H}^2_\pm$ the spaces of Hardy functions on the upper $(+)$ and lower (-) half planes, and by $\mathscr{H}^2_\pm|_{R^+}$ the spaces of the restrictions of the functions of $\mathscr{H}^2_\pm$ on the positive semi-axis, there is a one to one mapping $\theta_\pm$ [7] such that

$$\theta_\pm \mathscr{H}^2_\pm \mapsto \mathscr{H}^2_\pm \big|_{R^+} . \tag{13}$$

As a matter of fact, as our spaces of analytic functions, we take certain regular subspaces of $\mathscr{H}^2_\pm$. Let S be the space of all functions from R to C which are differentiable to all orders and that vanish at $\pm\infty$ faster than the inverse of any polynomial (Schwartz space). Then, let us consider the spaces $\Psi_\pm = \mathscr{H}^2_\pm \cap S$. We have the following relation between $\Psi_\pm$ and $\Phi_\pm$:

$$\theta_\pm \Psi_\pm \mapsto \Delta_\pm , \qquad V_\pm = \Omega_\pm U^{-1} \Delta_\pm \mapsto \Phi_\pm , \tag{14}$$

or, equivalently,

$$\Phi_\pm = V_\pm \theta_\pm \Psi_\pm . \tag{15}$$

The spaces of Hardy functions $\mathscr{H}^2_\pm$ are Hilbert spaces as subspaces of $L^2(R)$. Therefore, the norms in $\mathscr{H}^2_\pm$ and in $L^2(R)$ coincide. The mappings $\theta_\pm$ are one to one transformations from $\mathscr{H}^2_\pm$ into $L^2(R^+)$. In this sense $\theta_\pm$ are not unitary, from $L^2(R)$ onto $L^2(R^+)$, because for any $\varphi_\pm(E) \in \mathscr{H}^2_\pm$,

$$\|\theta_\pm \varphi_\pm\|_{L^2(R^+)} = \int_0^\infty |\varphi_\pm(E)|^2 \, dE < \int_{-\infty}^\infty |\varphi_\pm(E)|^2 \, dE = \|\varphi_\pm\|_{L^2(R)} < \infty . \tag{16}$$

In fact, $\varphi_\pm$ are boundary values of analytic functions and cannot be zero on the negative semi-axis unless they vanish identically.

Now, we can construct a new rigged Hilbert space which is given by $\Psi_\pm \subset \mathscr{H}^2_\pm \subset \Psi^\times_\pm$. The mappings $\theta_\pm$, induce two one to one mappings, $\theta^\times_\pm$, from $\Psi^\times_\pm$ onto $\Delta^\times_\pm$, by means of the identity

$$\langle \theta_\pm \varphi_\pm | \theta^\times_\pm G_\pm \rangle = \langle \varphi_\pm | G_\pm \rangle , \tag{17}$$

where $\varphi_\pm \in \Psi_\pm$ and $G_\pm \in \Psi^\times_\pm$. The mappings $\theta^\times_\pm$ are not extensions of $\theta_\pm$ because of the non-unitary of $\theta_\pm$. On the other hand, the unitary of $V_\pm$ allows us to extend them into the dual spaces by means of a similar formula:, i.e. if $\phi_\pm \in \Delta_\pm$ then $V_\pm \phi_\pm \in \Phi_\pm$ and if $F_\pm \in \Delta^\times_\pm$ then $V_\pm F_\pm \in \Phi^\times_\pm$ such that

$$\langle V_\pm \phi_\pm | V_\pm F_\pm \rangle = \langle \phi_\pm | F_\pm \rangle . \tag{18}$$

Thus, to any $\varphi \in \Phi_\pm$ corresponds an analytic function $\phi_\pm(E) \in \Psi_\pm$ and $\phi_\pm(E) = \theta^{-1}_\pm V^{-1}_\pm \varphi_\pm$. Therefore, the Gamow vectors can be represented as vectors in $\Psi_\pm$ as $(\theta^\times_+)^{-1} V^{-1}_+ | f_0 \rangle$ and $(\theta^\times_-)^{-1} V^{-1}_- | \tilde{f}_0 \rangle$. However, these formulas are unpractical to obtain the Gamow vectors in the new representation. In order to find them we shall use the definition of $|f_0\rangle$ and $|\tilde{f}_0\rangle$ and the Titchmarsh theorem [31–34] on Hardy functions. Then, we have

$$\langle \varphi_+ | f_0 \rangle = \phi_+^{\#}(z_R^*) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\phi_+^{\#}(E)}{E - z_R^*}\, dE \tag{19}$$

and

$$\langle \varphi_- | \tilde{f}_0 \rangle = \phi_-^{\#}(z_R) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\phi_-^{\#}(E)}{E - z_R}\, dE\,. \tag{20}$$

These formulas imply that

$$\left(\theta_+^{\times}\right)^{-1} V_+^{-1} | f_0 \rangle = \frac{-1}{2\pi i}\, \frac{1}{E - z_R^*} \tag{21}$$

and

$$\left(\theta_-^{\times}\right)^{-1} V_-^{-1} | \tilde{f}_0 \rangle = \frac{1}{2\pi i}\, \frac{1}{E - z_R}\,. \tag{22}$$

In this representation the Gamow vectors are square integrable, i.e., they belong to the Hilbert space $L^2(R)$. Therefore, we can define brackets and scalar products between them. We can also define energy averages on these vectors. This is, however, not an easy task. First of all, we must observe that in the new representation, the Hamiltonian H is given by $\hat{E} = V_{\pm}^{-1} H V_{\pm}$, where $V_{\pm} = V_{\pm}\theta_{\pm}^{-1}$. It is easy to show that $\hat{E}\phi_{\pm}(E) = E\phi_{\pm}(E)$, i.e., the multiplication operator on $L^2(R)$. This is why we do not add subscripts in $\hat{E}$. On $\mathscr{H}_{\pm}^2$, $\hat{E}$ is still symmetric but it is not self-adjoint (has different deficiency indices on $\mathscr{H}_{\pm}^2$). Its spectrum is purely continuous, simple and coincides with R. The definition of energy averages in this representation is hampered by the fact that

$$\hat{E}\,\frac{1}{E - z_R} = \frac{E}{E - z_R} \tag{23}$$

is not square integrable. This only means that the function $(E - z_R)^{-1}$ does not belong to the domain of the multiplication operator $\hat{E}$. However, since $\hat{E}$ can be extended by continuity to $\Psi^{\times}$, the identity (23) makes sense.

The representations (Eqs. (21) and (22)) of the Gamow vectors, when properly normalized, could be used now to define a mean value of the energy for these states. The normalization that we are going to use is the Hilbert space normalization, i.e., if we call

$$\psi^D = \alpha\, \frac{1}{E - z_R}\,, \qquad \psi^G = \alpha\, \frac{1}{E - z_R^*}\,, \tag{24}$$

then

$$\|\psi^D\|^2 = \alpha^2 \int_{-\infty}^{\infty} \frac{dE}{(E - z_R)^2 + (\Gamma/2)^2} = \alpha^2 \pi\,. \tag{25}$$

Therefore, $\|\psi^D\| = \|\psi^G\| = 1$ if $\alpha = 1/\sqrt{\pi}$.

Now, let us define the mean value of the energy on the decaying Gamow vector as

$$\langle \psi^D | \hat{E} | \psi^D \rangle \tag{26}$$

and let us evaluate its value. Since $\hat{E}\,\psi^D(E) = E\,\psi^D(E)$, we have that

$$\langle \psi^D | \hat{E} | \psi^D \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{E - z_R^*} \, \frac{E}{E - z_R} \, dE = \frac{2}{\pi \Gamma} \int_{-\infty}^{\infty} \frac{E\, dE}{\left(\dfrac{E - E_R}{\Gamma/2} \right)^2 + 1} \,. \tag{27}$$

The change of variables

$$x = \frac{E - E_R}{\Gamma/2} \tag{28}$$

transforms the last integral in (27) into

$$\frac{E_R}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} + \frac{\Gamma}{2\pi} \int_{-\infty}^{\infty} \frac{x\, dx}{x^2 + 1} \,. \tag{29}$$

The first integral in (29) has the value π. The second admits a Cauchy principal value equal to zero. Thus, we find

$$\langle \psi^D | \hat{E} | \psi^D \rangle = E_R \tag{30}$$

and

$$\langle \psi^G | \hat{E} | \psi^G \rangle = E_R \,. \tag{31}$$

We see that this definition of the energy average of Gamow vectors gives the same real value for both Gamow vectors and coincides with the resonant energy. In addition, due to the adopted normalization, we have that $\langle \psi^D | \psi^D \rangle = \langle \psi^G | \psi^G \rangle = 1$. Furthermore,

$$\langle \psi^G | \psi^D \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dE}{(E - z_R)^2} = 0 \,. \tag{32}$$

Analogously, $\langle \psi^D | \psi^G \rangle = 0$.

3.4. The averages are real in Berggren's interpretation

Berggren's approach to the mean value of the Hamiltonian on a Gamow state [17] can be formulated very similarly to Bohm's. Following [35,36], we shall not use Hardy functions to construct our Gelfand triplets. Instead, we consider here another triplet $\tilde{\xi} \subset \mathscr{H} \subset \tilde{\xi}^{\times}$ for which the space $\tilde{\xi}^{\times}$ consists of tempered ultra-distributions. A simple definition of these objects can be found in [35,36] and a complete account in [37,38]. Vectors in $\tilde{\xi}$ are entire analytic functions. Vectors in $\tilde{\xi}^{\times}$ are represented by pairs of analytic functions on the open upper and lower half planes, respectively. If we call $\psi_u(z)$ and $\psi_l(z)$ these functions, we can write

$$\psi(E) = \psi_u(E + i0) - \psi_l(E - i0) \,, \tag{33}$$

where $\psi_u(E+i0)$ and $\psi_l(E-i0)$ represent the boundary limits of $\psi_u(z)$ and $\psi_l(z)$ on the real axis, respectively. For any $\psi \in \tilde{\xi}^{\times}$ and any $z \in C - R$, we have

$$\psi(z) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(E)}{E-z}\, dE, \tag{34}$$

where we use in (34) the sign + or − for z on the upper or lower half plane, respectively,

$$\psi(z_R) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(E)}{E-z_R}\, dE, \qquad \psi(z_R^*) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(E)}{E-z_R^*}\, dE. \tag{35}$$

Now, we have a similar scheme to that presented in the previous section. The total Hamiltonian H is represented by the multiplication operator on $\tilde{\xi}$ and it can be extended as a continuous operator into the dual $\tilde{\xi}^{\times}$. The nuclear spectral theorem guarantees the existence of a complete set of eigenvectors $|E\rangle$ of H [39]. As eigenvectors, they obey the identity $H|E\rangle = E|E\rangle$. The Gamow vectors are now defined as[5]

$$|f_0\rangle = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{|E\rangle\, dE}{E-z_R}, \qquad |\tilde{f}_0\rangle = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{|E\rangle\, dE}{E-z_R^*}. \tag{36}$$

Within this scheme, the mean value of the Gamow vectors is defined as in the previous section and it gives the same results[6].

The coincidence between this last result for the mean value of the Gamow states and Bohm's one, presented in the last subsection, comes from a re-interpretation of Berggren's definition given in [17]. In fact, for a spherically symmetric potential and for an arbitrary value of the angular momentum l, we can write the normalized decaying Gamow vector as

$$|f_0\rangle = i\sqrt{\frac{2\Gamma}{\pi}} \int_0^{\infty} \sqrt{\frac{k}{m}}\; \frac{|k,\hat{k},l\rangle}{E(\boldsymbol{k})-z_R}\, dk, \tag{37}$$

where $k = |\boldsymbol{k}|$, $E(\boldsymbol{k}) = k^2/2m$ and $\hat{k}$ is the unit vector in the direction of $\boldsymbol{k}$. For the growing Gamow vector, we have

$$|\tilde{f}_0\rangle = -i\sqrt{\frac{2\Gamma}{\pi}} \int_0^{\infty} \sqrt{\frac{k}{m}}\; \frac{|k,\hat{k},l\rangle}{E(\boldsymbol{k})-z_R^*}\, dk. \tag{38}$$

Now, let A be an arbitrary observable. We can define the mean value of A on $|f_0\rangle$ as

$$\langle f_0|A|f_0\rangle = \frac{2\Gamma}{\pi} \sum_{l,l'} \int_0^{\infty} dk \int_0^{\infty} dk'\, \frac{\sqrt{kk'}}{m}\; \frac{\langle k',\hat{k}',l'|A|k,\hat{k},l\rangle}{\big(E(\boldsymbol{k}')-z_R\big)\big(E(\boldsymbol{k})-z_R^*\big)}. \tag{39}$$

If we replace A by H, we obtain, straightforwardly, the value E_R for this average.

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[5] The decaying Gamow vector is denoted in [35,36] as $|E_G^*\rangle$ and the growing Gamow vector as $|E_G\rangle$.

[6] In both cases, we can define the probability distribution associated to the Gamow states and it is given by (for the decaying Gamow vector) $P(E) = |\langle E|f_0\rangle|^2 = \frac{1}{\pi}\frac{\Gamma}{(E-E_R)^2+\Gamma^2}$. The same expression is found for the growing Gamow vector.

Instead, Berggren defines [17] this mean value as Real$\langle \tilde{f}_0 | A | f_0 \rangle$. As a matter of fact, one can easily show that

$$\langle f_0 | A | f_0 \rangle = \text{Real}\left\{ \langle \tilde{f}_0 | A | f_0 \rangle \right\} + o(\Gamma^2)\,, \tag{40}$$

which means that Berggren's approximation coincides with Bohm's to the first order in Γ.

4. Conclusions

In this paper we have compared different definitions of Gamow vectors. We have shown the equivalence between Bohm's and Berggren's definitions of the mean value of the Hamiltonian on a resonant state. Our main result, concerning this equivalence, is the realization of the average value of the Hamiltonian on a resonance as a real function which depends on both the real and the imaginary parts of the complex energy. This result is supported, mathematically, by a proper treatment of Gamow vectors in a rigged Hilbert space.

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