Boson mapping at finite temperature: An application to the thermo field dynamics

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The Holstein-Primakoff boson mapping at finite temperature and in the framework of the thermo field dynamics (TFD) is applied to treat a fermion-boson Hamiltonian. The interaction between pairs of fermions and bosons is constructed in a model which allows for a condensate. The evolution of the condensate as a function of the temperature is investigated. [S0556-2813(99)06208-1]

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I. INTRODUCTION

The thermo field dynamics (TFD) developed by Umezawa and co-workers [1] has been applied to describe a variety of physical situations [2–5]. The TFD shows that mean values computed in the standard statistical mechanics can be replaced by expectation values on a correlated vacuum. This vacuum has the structure of an exponential function of pairs (fermion or boson operators and their dual images) [6]. Gauge properties of the TFD and the symmetries associated to the TFD vacuum have been studied in Ref. [7]. As shown by Hatsuda [4] and by Klein and Marshalek [8] boson expansion techniques acting on the TFD image of a given Hamiltonian yield solutions which reproduce the exact results corresponding to the same Hamiltonian. Herewith we shall investigate the structure of the TFD image of the schematic Hamiltonian proposed by Schütte and Da Providencia [9]. This Hamiltonian includes the interaction between fermions and bosons. Its TFD image will be constructed by performing a boson expansion. The starting Hamiltonian has definite symmetries which are spontaneously broken by the TFD transformations. The subsequent boson mapping accommodates the generators of the original symmetry, in a larger space. In the first part of the present work we shall introduce the model and review the formalism of the thermo field dynamics. Next, a suitable boson mapping developed by Holstein and Primakoff (HP) [8] is extended to include thermal degrees of freedom. The spectrum of the transformed Hamiltonian is constructed by applying the random phase approximation (RPA). The results of the TFD+HP+RPA procedure are compared with the exact solution of the model both in the normal phase and in the condensed (or deformed) phase.

Details of the formalism are presented in the next section where the rules of the TFD and the main features of the boson expansion are introduced. The exact solution of the model of Schütte and Da Providencia [9] is described in Sec. II together with the approximated ones. The results of the thermal boson expansion, in the normal and deformed phases, are presented and discussed in detail in Sec. III. Conclusions are drawn in Sec. IV.

II. FORMALISM

A. Brief introduction to TFD

The thermo field dynamics is a theory where the statistical average of an operator is replaced by its vacuum expectation value, as done in quantum field theory [6]. In order to define a temperature dependent vacuum the doubling of the Hilbert space [2–4] is needed to preserve Gibbs probability distribution over physical states. The doubling of the Hilbert space, which should be accompanied by a definition of new operators which are acting on dual variables, is the new ingredient of the TFD which establishes the correspondence between statistical averages and expectation values [1]. In order to proceed with the formalism let us summarize the TFD basic rules in the following: (i) the Hilbert space is enlarged to accommodate physical and dual states (i.e., the TFD *tilde* states), (ii) the temperature dependent TFD vacuum is defined as an exponential function of physical and dual variables, (iii) the dynamics is then incorporated by introducing commutation relations between physical and dual variables and operators, (iv) true expectation values, of physical operators, are taken over the complete Hilbert space spanned by physical and dual (*tilde*) states, and (v) physical results are independent of *tilde* variables [6].

The TFD thermal vacuum $|0(\beta)\rangle$ is defined in such a way that the thermal expectation value of a given operator

$$\langle A\rangle = \mathrm{Tr}[\exp(-\beta H)A]/\mathrm{Tr}[\exp(-\beta H)], \tag{1}$$

is also written as the expectation value

$$\langle A\rangle = \langle 0(\beta)|A|0(\beta)\rangle, \tag{2}$$

where $\beta = 1/(kT)$ is the inverse temperature. The vacuum state $|0(\beta)\rangle$ is written [10]

$$|0(\beta)\rangle = [Z(\beta)]^{-1/2}\sum_m \exp(-\beta E_m/2)|m\rangle\otimes|\tilde{m}\rangle, \tag{3}$$

where $|m\rangle$ and $|\tilde{m}\rangle$ are the states of the physical and dual spaces.

Thermal identities [10] are satisfied by introducing a tilde conjugation rule which transforms operators from the original Fock space $\mathcal{I}$ to the dual one $\tilde{\mathcal{I}}$. Some useful TFD operations are [10]

$$\widetilde{AB} = \tilde{A}\tilde{B},$$

$$\widetilde{c_1 A + c_2 B} = c_1^{*}\tilde{A} + c_2^{*}\tilde{B},$$

$$\widetilde{A^{\dagger}} = \tilde{A}^{\dagger},$$

$$\tilde{\tilde{A}} = \pm A,$$

$$|0(\tilde{\beta})\rangle = |0(\beta)\rangle. \tag{4}$$

From the use of the TFD rules and the above relations, it can be shown that Schrödinger's equation in the original space translates into a Schrödinger's equation in the doubled space, with the Hamiltonian

$$\mathcal{H} = H - \tilde{H}. \tag{5}$$

There is a difference between TFD and the usual $(T=0)$ quantum mechanics. If the Hamiltonian H exhibits a dynamical symmetry $\mathcal{G} \subset \mathcal{S}$, where $\mathcal{S}$ is the algebra of all bilinear fermion or boson operators, the direct product group $\mathcal{G} \otimes \tilde{\mathcal{G}}$ is the dynamical symmetry group of the thermal Hamiltonian [8]. In general the thermal vacuum breaks this dynamical symmetry. As shown in [8] the minimal group $\mathcal{R}$, such that $\mathcal{G} \times \tilde{\mathcal{G}} \subseteq \mathcal{R} \subseteq \mathcal{S} \times \tilde{\mathcal{S}}$ and $|0(\beta)\rangle \in \mathcal{R}$, is the relevant symmetry group. In the following subsections we are going to illustrate the use of these concepts in dealing with the treatment of Schütte and Da Providencia model [9] at finite temperature.

B. The model of Schütte and Da Providencia

The model proposed by Schütte and Da Providencia [9] consists of N fermions moving in two single shells, each shell having 2Ω substates. The energy spacing between shells is fixed by the scale ω_f. Creation and annihilation operators of particles belonging to the upper shell are denoted by $c^\dagger_{2k}$ and c_{2k}, respectively, while for the lower shell these operators read $c^\dagger_{1k}$ and c_{1k}. Substates are denoted by the quantum number k. The fermions are coupled to an external boson field represented by the creation (annihilation) operator $b^\dagger$ (b) and by the energy ω_b.

The Hamiltonian reads [9]

$$H = \omega_f(t_0 + \Omega) + \omega_b b^\dagger b + G(t_+ b^\dagger + t_- b), \tag{6}$$

where G is the strength of the interaction in the particle-hole channel. The operators $t_\pm$ and t_0 are the generators of the algebra of the group SU(2) [11]. In terms of bilinear combinations of fermion operators these generators read

$$t_+ = \sum_k c^\dagger_{2k} c_{1k}, \quad t_- = (t_+)^\dagger,$$

$$t_0 = \frac{1}{2}(n + \overline{n}) - \Omega, \tag{7}$$

where

$$n = \sum_k c^\dagger_{2k} c_{2k}, \quad \overline{n} = \sum_k c_{1k} c^\dagger_{1k}, \tag{8}$$

are particle (n) and hole $(\overline{n})$ number operators.

The ''angular momentum'' P of the system is the difference between the number of bosonic and fermionic excitations, namely

$$P = b^\dagger b - \frac{1}{2}(n + \overline{n}). \tag{9}$$

This is a conserved quantity. The eigenvalues, L, of the operator P are integer numbers in the interval

$$-N \leqslant L \leqslant \infty. \tag{10}$$

This model can be interpreted as a simplified version of quarks in a two flavors and N colors representation interacting with a meson [12,13]. It was proposed by Schütte and Da Providencia [9] as an effective theory of baryons and it has solutions with unbroken (normal phase) and broken (deformed phase) P-symmetry [9].

In order to remind the reader about the model of [9] we shall briefly summarize the structure of the spectrum, both in the so-called normal and deformed phases and at zero temperature. In the normal phase the ground state is the eigenstate of P with $L=0$. In this phase the number of fermion pairs and the number of bosons are the same. This regime persists for small values of the coupling constant G. For large values of G the model exhibits a boson condensation and the ground state is an eigenstate of P with eigenvalue $L>0$. This is the deformed phase. For values $\omega_b > \omega_f$ and for intermediate values of G the ground state is an eigenvalue of P with $L<0$ and it represents a condensate of fermion pairs [9,14]. We shall now study the solutions of this model at finite temperature.

C. Mean field approximation at finite temperature

The mean field (MF) version of Hamiltonian of Eq. (6) can be written

$$H_{\mathrm{MF}} = E(\tau_0 + \Omega) + \omega_b \gamma^\dagger \gamma, \tag{11}$$

where

$$\tau_0 = \frac{1}{2}(\nu + \overline{\nu}) - \Omega, \tag{12}$$

and

$$\nu = \sum_k \alpha^\dagger_{2k} \alpha_{2k},$$

$$\overline{\nu} = \sum_k \alpha^\dagger_{1k} \alpha_{1k}, \tag{13}$$

are fermion number operators in the Hartree-Fock basis. The energy

$$E = \frac{\omega_f}{\cos\alpha}, \tag{14}$$

is the mean field value of the fermion energy. The boson operators $\gamma^\dagger$ and γ are the same as $b^\dagger$ and b except for the addition of vacuum expectation values ($\gamma^\dagger = b^\dagger - N^{1/2} b_0$, $\gamma = b - N^{1/2} b_0^*$) [9,12]. Following the TFD rules we add to these degrees of freedom their dual images $\tilde{\tau}_+$ and $\tilde{\gamma}^\dagger$ and the corresponding annihilation operators.

The thermal vacuum at mean field level is written [8]

$$|0(\beta)\rangle=[Z(\beta)]^{-1/2}\exp(-\beta H_{\mathrm{MF}}/2)\sum_m |m\rangle|\tilde{m}\rangle. \quad (15)$$

Since the operators τ_-, $(\tilde{\tau}_-)$ and γ $(\tilde{\gamma})$ do not annihilate the vacuum $|0(\beta)\rangle$, see for instance [8], we shall introduce the TFD thermal transformations T_- and a:

$$T_-=\sqrt{1-g_F(E/2)}\,\tau_-+2\sqrt{g_F(E/2)}\,\tau_0\tilde{\tau}_+, \quad (16)$$

$$a=\sqrt{1+g_B(\omega_b)}\,\gamma-\sqrt{g_B(\omega_b)}\,\tilde{\gamma}^\dagger. \quad (17)$$

The operators T_- (pair of fermions) and a (bosons) annihilate the thermal vacuum $|0(\beta)\rangle$. The factors $g_F(\epsilon_i)=1/(\exp(\beta\epsilon_i)+1)$ and $g_B(\epsilon_i)=1/(\exp(\beta\epsilon_i)-1)$ are Fermi-Dirac and Bose-Einstein occupation values, respectively. The transition from the normal to the deformed phase at mean field level is identified by performing the variation of H with respect to the vacuum expectation value of the boson number operator as an order parameter (see Appendix A). Self-consistency requires that [9]

$$1=\frac{x^2}{R}\tanh(R\beta'/2), \quad (18)$$

with $\cos\alpha=1/R$, $\beta'=(\omega_f/2)\beta$ and $x=G\sqrt{2\Omega/\omega_f\omega_b}$.

D. The boson expansion in the normal phase

What is usually referred to as the normal phase is the high temperature regime for which $\cos(\alpha)=1$ and $b_0=0$. In this case

$$H_{\mathrm{MF}}=\omega_f(t_0+\Omega)+\omega_b b^\dagger b. \quad (19)$$

One can rewrite the thermal vacuum as

$$|0(\beta)=\exp\left(-\omega_f\frac{\beta}{4}\right)\exp(t_+\tilde{t}_+)$$
$$\exp\left(-\omega_b\frac{\beta}{2}\right)\exp(b^\dagger\tilde{b}^\dagger)|0\rangle|\tilde{0}\rangle, \quad (20)$$

the thermal state can be rewritten as

$$|0(\beta)\rangle=\prod_k\,(u_F+v_F t_{+,k}\tilde{t}_{+,k})(u_B+v_B b^\dagger\tilde{b}^\dagger)|0\rangle|\tilde{0}\rangle. \quad (21)$$

We now use the fact that the thermal vacuum is unitary equivalent to the original vacuum and we define a new set of thermal operators of which $|0(\beta)\rangle$ is an eigenstate, namely

$$T_\pm=u_F t_\pm+2v_F t_0\tilde{t}_\mp,$$

$$\tilde{T}_\pm=u_F\tilde{t}_\pm+2v_F\tilde{t}_0 t_\mp,$$

$$T_0=u_F^2 t_0-v_F^2\tilde{t}_0-u_F v_F(t_+\tilde{t}_++t_-\tilde{t}_-),$$

$$\tilde{T}_0=u_F^2\tilde{t}_0-v_F^2 t_0-u_F v_F(t_+\tilde{t}_++t_-\tilde{t}_-), \quad (22)$$

where $T_-|0(\beta)\rangle=\tilde{T}_-|0(\beta)\rangle=0$, $T_0|0(\beta)\rangle=\tilde{T}_0|0(\beta)\rangle=-\Omega|0(\beta)\rangle$, and

$$u_F=1/\sqrt{\exp(-\beta\omega_f/2)+1},$$

$$v_F=\sqrt{1-u_F^2}. \quad (23)$$

These transformations between the operators have useful properties. They are nonlinear in the generators of SU(2) $\otimes$ SU(2) but the final form obeys an SU(2) $\otimes$ SU(2) algebra.

We will also need the inverse relations of Eq. (22), which are useful to calculate the thermal Hamiltonian

$$t_\pm=u_F T_\pm-2v_F T_0\tilde{T}_\mp,$$

$$\tilde{t}_\pm=u_F\tilde{T}_\pm-2v_F\tilde{T}_0 T_\mp,$$

$$t_0=u_F^2 T_0-v_F^2\tilde{T}_0+u_F v_F(T_+\tilde{T}_++T_-\tilde{T}_-),$$

$$\tilde{t}_0=u_F^2\tilde{T}_0-v_F^2 T_0+u_F v_F(T_+\tilde{T}_++T_-\tilde{T}_-). \quad (24)$$

These operators can be accommodated in a larger group [8] (see Appendix B). For the boson sector we shall introduce the transformations

$$b^\dagger=u_B a^\dagger+v_B\tilde{a},$$

$$\tilde{b}=v_B\tilde{a}+u_B a^\dagger, \quad (25)$$

and

$$a^\dagger=u_B b^\dagger-v_B\tilde{b},$$

$$\tilde{a}=-v_B\tilde{b}+u_B b^\dagger, \quad (26)$$

with

$$v_B=1/\sqrt{\exp(\beta\omega_b)-1},$$

$$u_B=\sqrt{1+v_B^2}. \quad (27)$$

The Hamiltonian reads

$$\begin{aligned}\mathcal{H}=&\,\omega_f(t_0-\tilde{t}_0)+\omega_b(a^\dagger a-\tilde{a}^\dagger\tilde{a})\\&+Gu_F u_B(T_+a^\dagger-\tilde{T}_+\tilde{a}^\dagger+\mathrm{H.c.})\\&+Gu_F v_B(T_+\tilde{a}-\tilde{T}_+a+\mathrm{H.c.})\\&+2Gv_F u_B(Y_{0-}a^\dagger-Y_{-0}\tilde{a}^\dagger+\mathrm{H.c.})\\&+2Gv_F u_B(Y_{0-}\tilde{a}-Y_{-0}a+\mathrm{H.c.}).\end{aligned} \quad (28)$$

The operators that give a nonzero result on the vacuum are the step operators T_+, $\tilde{T}_+$, Y_{0+}, Y_{+0} and Y_{++}, and the diagonal operators T_0, $\tilde{T}_0$ and Y_{00} (with eigenvalues $-\Omega$, $-\Omega$ and $-\Omega/2$) (see Appendix B). We now build an SU(4) irrep on the state $|0(\beta)\rangle$. Since we are considering states with 2Ω particles, the only irrep containing a state with the quantum numbers of $|0(\beta)\rangle$ is the irrep $\{2\Omega\}$. The boson mapping for this irrep can be constructed by using commutation techniques [8]. We introduce two bosons (B and $\tilde{B}$) for the Holstein-Primakoff mapping of the two SU(2) algebras. Using the commutation relations given in the Appendix B we can derive the form of all operators (see Appendix C). By applying the boson transformation and taking the leading order in the expansion $\mathcal{H}$ reads

$$\mathcal{H}=w_f(\eta^\dagger\eta-\tilde{\eta}^\dagger\tilde{\eta})+w_b(N_a-N_{\tilde{a}})$$
$$+g\sqrt{u_F^2-v_F^2}(a^\dagger\eta^\dagger+a\,\eta-\tilde{a}^\dagger\tilde{\eta}^\dagger-\tilde{a}\,\tilde{\eta}), \tag{29}$$

with

$$\eta^\dagger=\frac{1}{\sqrt{u_F^2-v_F^2}}(f_1B^\dagger-f_2\tilde{B}),$$

$$\tilde{\eta}=\frac{1}{\sqrt{u_F^2-v_F^2}}(f_1\tilde{B}-f_2B^\dagger), \tag{30}$$

where $g=G\sqrt{2\Omega}$ and

$$f_1=u_Fu_B-v_Fv_B\,,$$

$$f_2=u_Bv_F-u_Fv_B\,. \tag{31}$$

We shall diagonalize this Hamiltonian by applying the RPA formalism. The equations of motion are

$$[\mathcal{H},\Gamma_n^\dagger]=\omega_n\Gamma^\dagger,\quad [\mathcal{H},\tilde{\Gamma}_n^\dagger]=-\omega_n\tilde{\Gamma}^\dagger, \tag{32}$$

with

$$\Gamma^\dagger=X_1\eta^\dagger+X_2a^\dagger-Y_1\eta-Y_2a,$$

$$\tilde{\Gamma}^\dagger=X_1\tilde{\eta}^\dagger+X_2\tilde{a}^\dagger-Y_1\tilde{\eta}-Y_2\tilde{a}. \tag{33}$$

The RPA image of $\mathcal{H}$ is

$$\mathcal{H}=w(\Gamma^\dagger\Gamma-\tilde{\Gamma}^\dagger\tilde{\Gamma}), \tag{34}$$

where the eigenvalue ω is given by

$$\omega=\frac{1}{2}|w_f-w_b|+\frac{1}{2}\sqrt{(w_f+w_b)^2-4w_fw_bx^2\tanh(\beta\omega_f/4)}. \tag{35}$$

The expectation value of the symmetry operator is given by

$$\langle P\rangle=v_B^2-Nv_F^2\,. \tag{36}$$

E. The boson expansion in the deformed phase

As in the case of the normal phase we can write the thermal vacuum

$$|0(\beta)\rangle=\prod_k\,(u_F+v_F\tau_{+,k}\tilde{\tau}_{+,k})(u_B+v_B\gamma^\dagger\tilde{\gamma}^\dagger)|0\rangle|\tilde{0}\rangle, \tag{37}$$

where

$$\tau_+=\sum_k\,\alpha_{2k}^\dagger\alpha_{1k}^\dagger,\qquad \tau_-=\tau_+^\dagger\,,$$

$$\tau_0=\frac{1}{2}(\nu+\overline{\nu})-\Omega. \tag{38}$$

We can define a new set of operators, of which $|0(\beta)\rangle$ is an eigenstate, by introducing the expressions

$$T_\pm=u_F\tau_\pm+2v_F\tau_0\tilde{\tau}_\mp\,,$$

$$\tilde{T}_\pm=u_F\tilde{\tau}_\pm+2v_F\tilde{\tau}_0\tau_\mp\,,$$

$$T_0=u_F^2\tau_0-v_F^2\tilde{\tau}_0-u_Fv_F(\tau_+\tilde{\tau}_++\tau_-\tilde{\tau}_-),$$

$$\tilde{T}_0=u_F^2\tilde{\tau}_0-v_F^2\tau_0-u_Fv_F(\tau_+\tilde{\tau}_++\tau_-\tilde{\tau}_-), \tag{39}$$

where $T_-|0(\beta)\rangle=\tilde{T}_-|0(\beta)\rangle=0$ and

$$u_F=1/\sqrt{\exp(-\beta E/2)+1},$$

$$v_F=\sqrt{1-u_F^2}. \tag{40}$$

The same considerations, applied to the normal phase, are valid for the deformed phase concerning the group structure of these operators.

In the deformed phase the Hamiltonian of Eq. (6) is written

$$H=H_{\rm MF}+g_s(\tau_+\gamma+\tau_-\gamma^\dagger)+g_c(\tau_+\gamma^\dagger+\tau_-\gamma). \tag{41}$$

Finally, we shall work with the TFD Hamiltonian

$$\mathcal{H}=H_{\rm MF}-\tilde{H}_{\rm MF}+g_s(\tau_+\gamma+\tau_-\gamma^\dagger-\tilde{\tau}_+\tilde{\gamma}-\tilde{\tau}_-\tilde{\gamma}^\dagger)$$
$$+g_c(\tau_+\gamma^\dagger+\tau_-\gamma-\tilde{\tau}_+\tilde{\gamma}^\dagger-\tilde{\tau}_-\tilde{\gamma}), \tag{42}$$

and after some algebra it reads

$$\mathcal{H}=w_f(\eta^\dagger\eta-\tilde{\eta}^\dagger\tilde{\eta})+w_b(N_a-N_{\tilde{a}})+g_c\sqrt{u_F^2-v_F^2}(a^\dagger\eta^\dagger+a\,\eta$$
$$-\tilde{a}^\dagger\tilde{\eta}^\dagger-\tilde{a}\,\tilde{\eta})+g_s\sqrt{u_F^2-v_F^2}(a^\dagger\eta+a\,\eta^\dagger-\tilde{a}^\dagger\tilde{\eta}-\tilde{a}\,\tilde{\eta}^\dagger), \tag{43}$$

with

$$\eta^\dagger=\frac{1}{\sqrt{u_F^2-v_F^2}}(f_1B^\dagger-f_2\tilde{B}),$$

$$\tilde{\eta}=\frac{1}{\sqrt{u_F^2-v_F^2}}(f_1\tilde{B}-f_2B^\dagger), \tag{44}$$

where $g_c = g[\cos(\alpha/2)]^2$, $g_s = -g[\sin(\alpha/2)]^2$ and

$$f_1 = u_F u_B - v_F v_B,$$

$$f_2 = u_B v_F - u_F v_B. \tag{45}$$

We shall diagonalize the Hamiltonian by applying the RPA formalism. The equations of motion are

$$[\mathcal{H}, \Gamma_n^\dagger] = \omega_n \Gamma^\dagger, \quad [\mathcal{H}, \tilde{\Gamma}_n^\dagger] = -\omega_n \tilde{\Gamma}^\dagger, \tag{46}$$

with

$$\Gamma^\dagger = X_1 \eta^\dagger + X_2 a^\dagger - Y_1 \eta - Y_2 a,$$

$$\tilde{\Gamma}^\dagger = X_1 \tilde{\eta}^\dagger + X_2 \tilde{a}^\dagger - Y_1 \tilde{\eta} - Y_2 \tilde{a}, \tag{47}$$

and

$$\mathcal{H} = w(\Gamma^\dagger \Gamma - \tilde{\Gamma}^\dagger \tilde{\Gamma}). \tag{48}$$

The two solutions for ω are $\omega = 0$ and

$$\omega = \sqrt{E^2 + \omega_b^2 - 2\omega_b \omega_f}. \tag{49}$$

The expectation value of the symmetry operator is given by

$$\langle P \rangle = v_B^2 + N\left(b_0^2 - \frac{1}{2} + \frac{1}{2}\cos\alpha(1 - 2v_F^2)\right). \tag{50}$$

F. The exact partition function

In order to evaluate the grand partition function [15], we have to calculate the eigenvalues $E_{L,m}^T$ and the multiplicity of the irreducible representations Γ_T for different particle numbers, namely $0 < N \leqslant 4\Omega$.

The physical space is spanned by the vectors

$$\{|\epsilon_1 k_1, \epsilon_2 k_2, \ldots, \epsilon_n k_n\rangle \otimes |l\rangle, \tag{51}$$

$$\epsilon_i \in \{1,2\}, k_i \in \{1,\ldots,2\Omega+1\}, i \in \{1,2,\ldots,n\},$$

$$n \in \{1,2,\ldots,4\Omega\}, l \in \{0,\ldots,\infty\}\},$$

where $|\epsilon_1 k_1, \epsilon_2 k_2, \ldots, \epsilon_n k_n\rangle$ represents the fermionic subspace and $|l\rangle$ the bosonic one. ϵ_i is the index corresponding to levels, k represents substates and i reads for the partition with i particles and n is the particle number of the configuration. The index l corresponds to the number bosons in a particular state.

For the fermionic subspace the number of vectors associated to a system with two levels, each of them with 2Ω substates and with a number of particles varying from 1 to 4Ω, is equal to $2^{4\Omega}$. The bosonic subspace has an infinite dimension.

The fermionic subspace can be decomposed into invariant and irreducible subspaces. A particular distribution of given number of particles on two degenerate levels can be characterized by numbers ν_1 and ν_2, i.e., ν_1 is the number of sublevels which are occupied by particles in both the lower and the upper levels, ν_2 is the number of sublevels which are unoccupied in the lower and upper levels. The quasispin T of the state is determined by the distribution the particles on the 2τ sublevels, where $2\tau = 2\Omega - \nu_1 - \nu_2$. The number of particles in this configuration is $n = 2(\tau + \nu_1)$. Let us call $\Gamma_{k_1, k_2, \ldots, k_{2(\tau+\nu_1)}}$ the subspace of states with ν_1 occupied and ν_2 unoccupied sublevels. The dimension is $2^{2\tau}$. They are $(2\Omega)!/((2\tau)!\nu_1!\nu_2!)$ different subspaces $\Gamma_{k_1, k_2, \ldots, k_{2(\tau+\nu_1)}}$. Each of these subspaces can be decomposed into irreducible ones with multiplicities

$$g_k^\tau = \frac{(2\tau)!}{k!(2\tau - k)!} - \frac{(2\tau)!}{(k-1)!(2\tau - k + 1)!}.$$

The exact grand partition function can be written

$$\mathcal{Z}(\beta) = \sum_{\tau \nu_1 \nu_2} \frac{2\Omega!}{(2\tau)!\nu_1!\nu_2!} \times \sum_k g_k^\tau \sum_{L,m} \exp[-\beta(E_{L,m}^{\tau-k} - 2\mu(\tau + \nu_1))]. \tag{52}$$

The expressions for the average energy and the average of the operator P are given by

$$\langle H \rangle = \sum_{\tau \nu_1 \nu_2} \frac{2\Omega!}{(2\tau)!\nu_1!\nu_2!} \times \sum_k g_k^\tau \sum_{L,m} E_{L,m}^{\tau-k} \exp[-\beta(E_{L,m}^{\tau-k} - 2\mu(\tau + \nu_1))], \tag{53}$$

and

$$\langle P \rangle = \sum_{\tau \nu_1 \nu_2} \frac{2\Omega!}{(2\tau)!\nu_1!\nu_2!} \times \sum_k g_k^\tau \sum_{L,m} L \exp[-\beta(E_{L,m}^{\tau-k} - 2\mu(\tau + \nu_1))]. \tag{54}$$

In the above formulas μ is the Lagrange multiplier which enforces the number of particles constraint and $E_{L,m}^{\tau-k}$ represents the exact eigenvalues corresponding to $\tau - k$ particles.

III. RESULTS AND DISCUSSIONS

In order to compare the validity of the approximations described in the previous section we have calculated exact results corresponding to the Hamiltonian of Schütte and Da Providencia, using the equations presented in Sec. II F. We have considered the case $N = 30$ particles moving in two levels, each of them with $\Omega = 15$. The Hamiltonian has been diagonalized in each subspace corresponding to a certain partition of the particle number. The exact grand partition function is afterwards calculated by the sum over all partitions and all allowed values of the symmetry eigenvalue L, actual

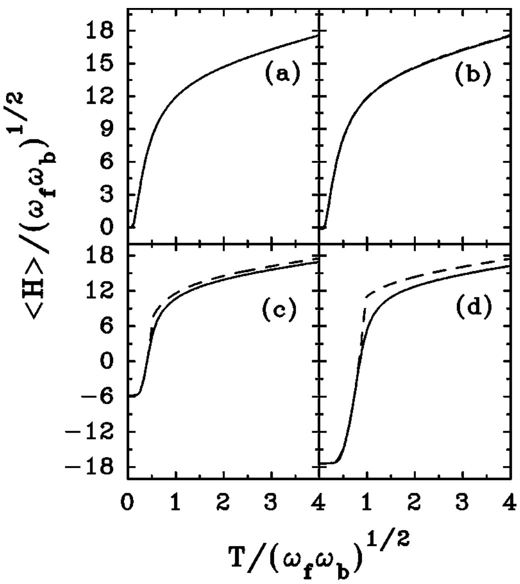

FIG. 1. Mean value of the energy as a function of the temperature, corresponding to the Hamiltonian of Ref. [9]. Cases (a), (b), (c), and (d) are the results obtained with the reduced coupling [see Eq. (18)] $x=0$, 0.5, 1.5, and 2, respectively. Solid lines represent exact results and dashed lines are the results obtained by using the TFD procedure as described in the text.

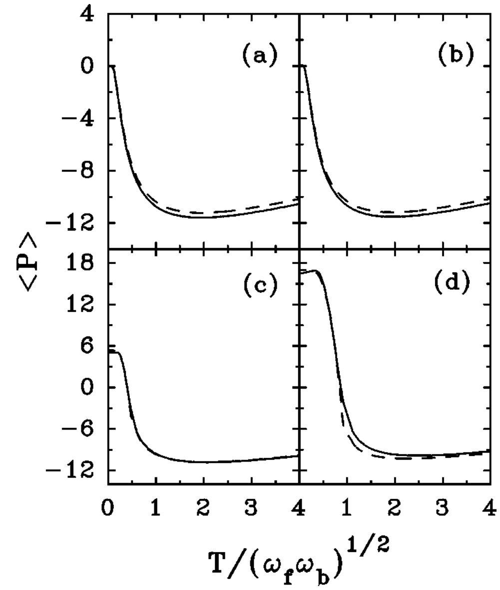

FIG. 2. Mean value of the symmetry P, see Eq. (9), as a function of the temperature. Cases (a), (b), (c), and (d) show the results obtained with the reduced coupling $x=0$, 0.5, 1.5, and 2, respectively. Solid lines represent exact results and dashed lines are the results given by the TFD.

calculations where performed for values of L up to $L=200$. The approximations described in the previous section have been implemented by performing the RPA diagonalization after applying the boson expansion in the TFD basis. Thus the results obtained in this way are limited by the cutoff implied by the RPA expansion. The comparison between exact and approximated results for the expectation value of the Hamiltonian are shown in Fig. 1. Results corresponding to the value of the symmetry, P, are shown in Fig. 2. The agreement between exact and approximated results is verified on a large domain of values of the reduced coupling x. It works both in the normal and in the deformed phases. As it is well known the RPA approximation, at finite temperatures, has a phase transition point. The results shown in Fig. 3 indicates that the critical temperature depends upon the strength of the interaction. These results also indicate that the normal and deformed phases are separated if the reduced coupling strength x is fixed above or bellow the critical value $x=1$. Temperature dependent effects at the RPA level of approximation seem to be rather small for values of $x<1$.

From the curves shown in Fig. 1, it is seen that a linear regime is obtained for reduced temperatures larger than one, while a fast increase of the mean value of the energy is observed for values of $T/(\omega_f\omega_b)^{(1/2)}<1$. The correspondence between these results and the mean value of the symmetry operator P, displayed in Fig. 2, clearly shows the structure of the different phases. The transition between the boson and the fermion condensates is also depending on the value of the reduced coupling and it is obtained for values of the reduced temperature $T/(\omega_f\omega_b)^{(1/2)}<1$ [see cases (c) and (d) of Fig. 2]. Concerning the RPA results, see Fig. 3, the transition to a normal phase at reduced temperatures $T/(\omega_f\omega_b)^{(1/2)}>1$ is obtained, as it has been suggested by Klein and Marshaleck [8]. The TFD results shown in Fig. 1(c) and Fig. 1(d) reflect the change in the mean field induced by strong interactions. It is seen that the bulk of the correlation energy is given by the RPA value, as shown in Fig. 3, until thermal excitations produce the screening of the interactions between fermions and bosons. Naturally, this transition from one mean-field (interacting system of fermions and bosons) to another (a noninteracting system of two-components) is a feature of the approximate solution which has, at any point of the expansion in terms of the coupling

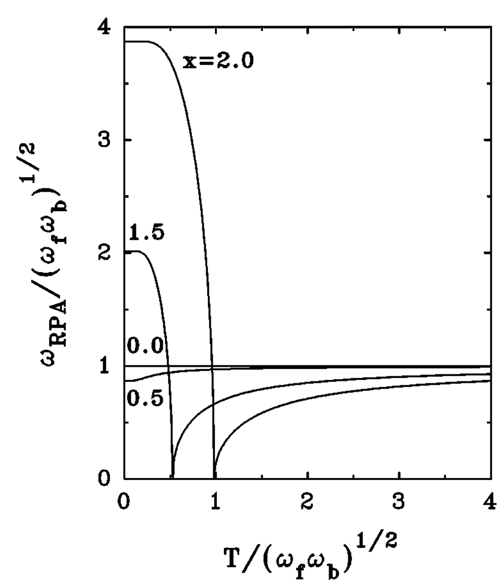

FIG. 3. Temperature dependence of the RPA frequency. The values of the reduced coupling strength x are indicated on the curves. The RPA phase transition appears for x larger than 1.

constants, a lesser number of contributions than the exact result.

IV. SUMMARY AND OUTLOOK

In this paper we have performed a boson expansion of the TFD image of the Hamiltonian introduced by Schütte and Da Providencia. Following the general steps of the TFD we have shown that the RPA treatment of the Hamiltonian produces results which are very similar to the exact ones. In this way the conclusions advanced by Klein and Marshaleck in their fundamental paper on boson expansions [8] proved to be appropriate also when the boson expansions are applied to describe interactions at finite temperatures. Since the agreement between the exact results and the results of the boson mapping at finite temperature does not appear to be accidental we may conclude that the use of TFD in conjunction with the boson mapping and the RPA could be very well applied to study more sophisticated Hamiltonians.

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APPENDIX A

The deformed solution is obtained by using the following transformations of the boson and fermion operators:

$$b^\dagger = \gamma^\dagger + N^{1/2} b_0,$$

$$c_{2k}^\dagger = \cos\left(\frac{\alpha}{2}\right)\alpha_{2k}^\dagger - \sin\left(\frac{\alpha}{2}\right)\alpha_{1k},$$

$$c_{1k}^\dagger = \sin\left(\frac{\alpha}{2}\right)\alpha_{2k}^\dagger + \cos\left(\frac{\alpha}{2}\right)\alpha_{1k}.$$

In view of this we can redefine the number operator and the raising and lowering operators for quasiparticles:

$$\nu = \sum_k \alpha_{2k}^\dagger \alpha_{2k}, \quad \overline{\nu} = \sum_k \alpha_{1k}^\dagger \alpha_{1k},$$

$$\tau_+ = \sum_k \alpha_{2k}^\dagger \alpha_{1k}^\dagger, \quad \tau_- = (\tau_+)^\dagger,$$

$$\tau_0 = \frac{1}{2}(\nu + \overline{\nu}) - \Omega.$$

In this representation the angular momentum reads

$$P = N\left(b_0^2 - \sin^2\left(\frac{\alpha}{2}\right)\right) + \gamma^\dagger\gamma - N\cos(\alpha)(\tau_0 + \Omega) + N^{1/2} b_0(\gamma^\dagger + \gamma) + \sin(\alpha)(\tau_+ + \tau_-),$$

and the Hamiltonian is written

$$H = N(\omega_b b_0^2 + \omega_f \sin^2(\alpha/2) - g\sin(\alpha)) + (\omega_f\cos(\alpha) + 2gb_0\sin(\alpha))(\tau_0 + \Omega) + \omega_b(\gamma^\dagger\gamma) + (gb_0\cos(\alpha) - \omega_f\sin(\alpha))(\tau_+ + \tau_-) + \left(-\frac{N}{2}G\sin(\alpha)\tanh(\beta E/2) + N^{1/2} b_0\omega_b\right)(\gamma^\dagger + \gamma) + g_s(\tau_+\gamma + \gamma^\dagger\tau_-) + g_c(\tau_+\gamma^\dagger + \gamma\tau_-) + G\sin(\alpha)\frac{1}{2}(\nu + \overline{\nu})(\gamma^\dagger + \gamma),$$

with $g_c = GN^{1/2}\cos^2(\alpha/2)$ and $g_s = -GN^{1/2}\sin^2(\alpha/2)$.

After applying the transformation of Eq. (39) and minimizing $H_{00} = \langle 0(\beta)|H|0(\beta)\rangle$ with respect to b_0 and α, the Hamiltonian is written

$$H_{11} = E\frac{1}{2}(\nu + \overline{\nu}) + \omega_b\gamma^\dagger\gamma,$$

$$H_{20} = 0,$$

where $E = \omega_f/\cos(\alpha)$, $\cos\alpha = 1/R$, $b_0^2 = (w_f/4w_b)x^2(1 - 1/R^2)$, and $1 = (x^2/R)\tanh(R\beta'/2)$ with $\beta' = (\omega_f/2)\beta$.

APPENDIX B

Let us define

$$Y_{\alpha\beta} = T_\alpha \tilde{T}_\beta,$$

with $\alpha,\beta = 0,\pm$.

The 15 operators T_α, $\tilde{T}_\alpha$ and $Y_{\alpha\beta}$ close the SU(4) algebra. The commutation rules of these operators in the Cartesian form are

$$[T_i, T_j] = i\epsilon_{ijk}T_k, \quad [\tilde{T}_i, \tilde{T}_j] = i\epsilon_{ijk}\tilde{T}_k,$$

$$[\tilde{T}_i, T_j] = 0,$$

$$[T_i, Y_{jk}] = i\epsilon_{ijl}Y_{lk}, \quad [\tilde{T}_i, Y_{jk}] = i\epsilon_{ikl}Y_{jl},$$

$$[Y_{ij}, Y_{kl}] = i\epsilon_{ikm}T_m\delta_{jl} + i\epsilon_{jlm}\tilde{T}_m\delta_{ik}.$$

APPENDIX C

The boson mapping for the operators of Appendix B is constructed by commutator techniques [8]. We introduce two bosons (B and $\tilde{B}$) for the Holstein-Primakoff mapping of the two SU(2) algebras, and one extra boson ($C=\tilde{C}$) that decreases both T and $\tilde{T}$ by one. The most general forms for the transformed operators are

$$T_+ = B^\dagger(2\Omega - 2N_C - N_B)^{1/2}, \quad T_- = T_+^\dagger,$$

$$T_0 = -\Omega + N_C + N_B;$$

$$\tilde{T}_+ = \tilde{B}^\dagger(2\Omega - 2N_C - N_{\tilde{B}})^{1/2}, \quad \tilde{T}_- = \tilde{T}_+^\dagger,$$

$$\tilde{T}_0 = -\Omega + N_C + N_{\tilde{B}},$$

$$Y_{++} = B^\dagger \tilde{B}^\dagger r(N_1) r(N_2) b_1(N_C) + C^\dagger r(N_1) r(N_1+1) \times r(N_2) r(N_2+1) b_2(N_C) + B^{\dagger 2}\tilde{B}^{\dagger 2} C b_3(N_C),$$

$$Y_{0+} = \tilde{B}^\dagger(-\Omega + 2N_C + N_B) r(N_2) b_1(N_C) + \tilde{B} C^\dagger r(N_1) r(N_2) r(N_2+1) b_2(N_C) - B^\dagger \tilde{B}^{\dagger 2} C r(N_1 - 1) b_3(N_C),$$

$$Y_{-+} = B\tilde{B}^\dagger r(N_1 - 1) r(N_2) b_1(N_C) - B^2 C^\dagger r(N_2) r(N_2+1) \times b_2(N_C) - \tilde{B}^{\dagger 2} C r(N_1 - 2) r(N_1 - 1) b_3(N_C),$$

$$Y_{00} = (\Omega - 2N_C - N_B)(\Omega - 2N_C - N_{\tilde{B}}) b_1(N_C) + B\tilde{B} r(N_1) r(N_2) b_2(N_C) + B^\dagger \tilde{B}^\dagger C r(N_1 - 1) \times r(N_2 - 1) b_3(N_C),$$

$$r(N) = (2\Omega - N)^{1/2},$$

$$N_1 = 2N_C + N_B, \quad N_2 = 2N_C + N_{\tilde{B}},$$

$$b_1(N_C) = \frac{\Omega + 1}{2(\Omega - N_C)(\Omega - N_C + 1)},$$

$$b_2(N_C) = b_3(N_C + 1);$$

$$b_3(N_C) = \frac{1}{2(\Omega - N_C + 1)} \times \left[\frac{2\Omega - N_C + 2}{(2\Omega - N_C + 3)(2\Omega - N_C + 1)}\right]^{1/2}.$$

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