Gamow States in a Rigged Hilbert Space *

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Summary. The space of analytical test fuctions rapidly decreasing on the real axis (i.e: Schwartz test functions on the real axis), is used to construct the Rigged Hilbert Space (RHS) where Resonant Gamow States (GS) are defined starting from Dirac's formula. It is shown that the expectation value of a self-adjoint operator acting on a GS is real.

The treatment of the continuum and the inclusion of decaying states in the definition of the nuclear response is a long-standing problem [1, 2, 3]. The inclusion of resonant states in the one-body Green Function has been studied years ago by Tore Berggren [4, 5]. Lately, the use of these states to calculate one-particle and collective excitations in finite nuclei has been proposed by Liotta et al. [6].

Several methods have been developed in connection with the treatment of GS [7, 8, 9]. The equivalence between some of these methods and the correspondence between Bergreen's and Mittag-Lefler's representations have been explored in dealing with the use of GS in nuclear structure problems [6]. Mathematical properties of GS, in the framework of the Hamiltonian formalism, have been studied by Sudarshan and collaborators [10]. Bohm et al. [11, 12] have shown that the RHS is a suitable framework to describe idealized resonances as generalized eigenvectors of a self-adjoint Hamiltonian with complex eigenvalues. The overlap between GS and wave packets of the Breit-Wigner form has been discussed by Romo [8] by using techniques of analytic continuation. The possibility of defining expectation values of operators in a resonant state has been studied by Tore Berggren in a recent work [13].

In this talk we shall show some results concerning the calculation of expectation values on resonant states [13]. At variance with the usually adopted mathematical formalism [7, 12] we shall use the concepts of tempered ultradistributions and Gelfand's triplets [14]. In the following, only the aspects of the derivation which are relevant to validate Berggren's approximation will be shown. The mathematical details of the formalism are given in [15].

The space of analytical functionals ξ' (tempered ultradistributions) is the minimal space whose Fourier anti-transform accomodates real exponential functions as distributions. This space is the dual of the space of analytical test functions ξ. Together with the Hilbert space $\mathcal{H}$ one can construct the RHS or Gelfand's triplet (GT) [14, 15] $\xi \subset \mathcal{H} \subset \xi'$. In this RHS a linear and

* Dedicated to the late Professor Tore Berggren

symmetric operator A acting on ξ, which admits a self-adjoint prolongation $\bar{A}$ acting on $\mathcal{H}$, has a complete set of eigen-functionals on ξ' with real generalized eigenvalues [14]. Let us introduce the GT $(\tilde{\xi}, \mathcal{H}, \tilde{\xi}')$ which is related to $(\xi, \mathcal{H}, \xi')$ by Fourier transforms. The Schwartz Space $\mathcal{S}'$ of tempered distributions is included in ξ' and in $\tilde{\xi}'$ $(\mathcal{S}' \in \xi')$. The extension to ξ' of Dirac's formula is given by [15]

$$\hat{\psi}_c(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1}{t - z} \hat{\psi}(t)\, dt, \tag{1}$$

where $\hat{\psi}(t) = \hat{\psi}_c(t+i0) - \hat{\psi}_c(t-i0)$. In addition to $(\tilde{\xi}, \mathcal{H}, \tilde{\xi}')$ and $(\xi, \mathcal{H}, \xi')$ it exists the GT $(\xi_a, \mathcal{H}_a, \xi_a')$ which admits the definition of the position operator $\hat{x}$ acting on $\mathcal{H}_a$. If $|x> \in \xi_a'$ then $< x'|x >= \delta(x - x')$ and $< x'|\hat{x}|x >= x\delta(x - x')$.

The relations $|\phi > \in \xi_a \Leftrightarrow < x|\phi >= \phi(x) \in \tilde{\xi}$, $|\varphi > \in \mathcal{H}_a \Leftrightarrow < x|\varphi >= \varphi(x) \in \mathcal{H}$ and $|\psi > \in \xi_a' \Leftrightarrow < x|\psi >= \psi(x) \in \tilde{\xi}'$ represent Dirac's formalism of Quantum Mechanics in a RHS [16]. Let us introduce a self-adjoint operator H, acting on $\mathcal{H}$, with the eigenstates (eigenvalues) given by $H|E_n >= E_n|E_n >$ (for $n \in \mathcal{N}$) and $H|E >= E|E >$ (for $E_0 < E < E_1$). Thus from Eq.(1) one can write

$$(\hat{\psi}(E_G))^* = \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{{E_G}^* - E} (\hat{\psi}(E))^*\, dE\ , \tag{2}$$

with $E_G = E_D + i\Gamma \quad \Gamma > 0$. In Dirac's notation one has

$$(\hat{\psi}(E_G))^* = \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{{E_G}^* - E} < \psi|E > \ dE\ . \tag{3}$$

We can now define

$$|{E_G}^* >= \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{{E_G}^* - E} |E > \ dE\ . \tag{4}$$

Then $\hat{\psi}(E_G) =< E_G|\psi >$ and $(\hat{\psi}(E_G))^* =< \psi|{E_G}^* >$. The state $|{E_G}^* >$ is by definition a GS eigenstate of H.

The states $|{E_G}^* >$ are normalizable. The diagonal matrix element of H between GS is given by the expression

$$< E_G|H|E_G^* >= E_D + \frac{\Gamma}{2}\frac{\ln\left[\frac{(E_1-E_D)^2+\Gamma^2}{(E_0-E_D)^2+\Gamma^2}\right]}{\left[\arctan\left(\frac{E_1-E_D}{\Gamma}\right)-\arctan\left(\frac{E_0-E_D}{\Gamma}\right)\right]} \quad (5)$$

The probability distribution associated to a GS is given by

$$P(E) = |< E|E_G^* >|^2$$

$$= \frac{\Gamma}{(E-E_D)^2+\Gamma^2} \cdot \frac{1}{\left[\arctan\left(\frac{E_1-E_D}{\Gamma}\right)-\arctan\left(\frac{E_0-E_D}{\Gamma}\right)\right]} \quad (6)$$

as proposed in [8, 11].

For a self-adjoint operator A, which is acting on $\mathcal{H}_a$, the expectation value of A between GS

$$< E_G|A|E_G{}^* >= \int_{-\infty}^{+\infty} < E_G|\lambda > \lambda\, d\sigma_a(\lambda) < \lambda|E_G{}^* > \ , \quad (7)$$

is real since $< E_G|\lambda >= (< \lambda|E_G^* >)^*$.

Following Berggren's notation [13] the GS state can be defined by

$$|E_G{}^* >= \frac{\sqrt{\Gamma}}{i\sqrt{\pi/2}}\int_0^{+\infty}\frac{|E(\vec{k}),\hat{k},l>}{E_G{}^*-E}\, dE\ , \quad (8)$$

and in the impulse representation it is written as

$$|E_G{}^* >= \frac{\sqrt{\Gamma}}{i\sqrt{\pi/2}}\int_0^{+\infty}\sqrt{\frac{k}{m}}\,\frac{|k,\hat{k},l>}{E_G{}^*-E(\vec{k})}\, dk\ . \quad (9)$$

Consequently, for the expectation value of A one has the expression

(10)

$$< E_G|A|E_G{}^* >= \frac{2\Gamma}{\pi}\sum_{l,l'}\int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m}\frac{< k',\hat{k}',l'|A|k,\hat{k},l >}{(E(\vec{k}')-E_G)(E(\vec{k})-E_G{}^*)}\ .$$

We can now compare the result provided by the present method and by Berggren's conjecture, namely: $< A >= Re < E_G{}^*|A|E_G{}^* >$, where

(11)

$$< E_G{}^*|A|E_G{}^* >= \frac{2\Gamma}{\pi}\sum_{l,l'}\int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m}\frac{< k',\hat{k}',l'|A|k,\hat{k},l >}{(E(\vec{k}')-E_G{}^*)(E(\vec{k})-E_G{}^*)}\ .$$

The relation between the above equations can be expressed as

$$< A >=< E_G|A|E_G{}^* >= Re < E_G{}^*|A|E_G{}^* > -$$

$$\frac{2i\Gamma^2}{\pi}\sum_{l,l'}\int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m}\left[E(\vec{k}) - E(\vec{k}')\right] \frac{< k', \hat{k}', l'|A|k, \hat{k}, l >}{|E(\vec{k}') - E_G|^2 |E(\vec{k}) - E_G|^2} +$$

$$\frac{4\Gamma^3}{\pi}\sum_{l,l'}\int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m} \frac{< k', \hat{k}', l'|A|k, \hat{k}, l >}{|E(\vec{k}') - E_G|^2 |E(\vec{k}) - E_G|^2} . \tag{12}$$

It means that the result obtained in [13] is valid at leading order in Γ and that, in general, $< E_G|A|E_G{}^* > \neq Re < E_G{}^*|A|E_G{}^* >$.

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