Comparative study on the validity of the renormalized random-phase approximation

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The consistency of the renormalized random-phase approximation (RRPA), to treat correlations nearby a phase transition, is analyzed. The results of the RRPA and its quasiparticle version (RQRPA) for monopole and quadrupole particle-hole excitations and pairing vibrational modes, are compared with exact results and with results of the conventional quasiparticle random-phase approximation (QRPA). The calculations are performed in schematic models. [S0556-2813(98)05806-3]

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I. INTRODUCTION

The quasiparticle random-phase approximation (QRPA), introduced long ago by Baranger [1], has been extensively used to describe two-body correlations in open-shell nuclei. The various applications of the method to low-energy nuclear structure physics can be found in textbooks [2,3] and in review articles [4].

The basic assumptions of the method are: (i) the definition of a quasiparticle mean field to account for pairing correlations, and (ii) the inclusion of two-body residual interactions between quasiparticles. These assumptions have been tested rather successfully in dealing with the microscopic description of like-particle pair excitations in open shells [5]. The first applications of the QRPA to describe unlike-particle pair excitations (proton-neutron pairs) were performed by Hableib and Sorensen [6]. The interest in the method was renewed by the study of the effects produced upon nuclear double-β decay observables by renormalized two-particle (proton-neutron) interactions [7,8].

Several attempts to prevent the so-called collapse of the proton-neutron (pn) QRPA approximation have been reported after the results of Vogel and Zirnbauer [7] were published. For a review of some of these approaches see [9,10]. Particularly, the method developed by Hara [11] and applied to the Lipkin model by Catara *et al.* [12] has been extended to treat nuclear double-β decay matrix elements by Toivanen and Suhonen [13]. Difficulties related to this approach have been reported in [14,15].

The question about the validity of the RQRPA in realistic cases cannot always be answered by a direct comparison with exact (shell-model) results [16]. However, the advantages and/or disadvantages of the RQRPA can be investigated in solvable models for which exact solutions are known [17].

It is the aim of the present work to compare RQRPA, QRPA and exact results for particle-hole monopole and quadrupole excitations and monopole pairing vibrational modes, both in normal and superfluid phases. Since some of the unknown nuclear-structure elements of the nuclear double-β decay problem are not present in this case, i.e., the uncertainties associated with the treatment of ground-state correlations induced by unlike (proton-neutron) pairs [6], we hope to extract more clear conclusions about the suitability of the RQRPA approach. We would like to concentrate on the ability of the RQRPA to describe correlations near the QRPA (RPA) phase transition [18].

The calculations were performed for schematic models where the single-particle energy spacing and the effective degeneracy of the single-particle levels are fixed. The coupling constants of the residual two-body interactions were varied freely around critical values, i.e., the values of the RPA collapse, for each model Hamiltonian. Due to the schematic nature of the interactions self-consistent determinations of mean-field properties were not implemented. Self-energy corrections to single-particle (quasiparticle energies) and phonon energies were not considered, either. The contributions of the interactions to RPA and renormalized RPA (RRPA) processes were calculated at leading order in the expansion parameter Ω, which is the degeneracy of the single-particle levels. Other effects, like the rearrangement of mean-field values due to couplings with phonons and the renormalization of particle-vibration couplings due to phonon self-energies, were not considered in this work since they contribute at lower order in the $1/\Omega$ expansion [17]. Limitations due to these approximations will be discussed with reference to schematic and realistic calculations reported in the literature [19–24].

A brief review of the formalism is presented in Sec. II, where the QRPA and RQRPA methods are introduced to treat exactly solvable Hamiltonians. In Sec. III the results of the calculations are discussed. Conclusions are drawn in Sec. IV.

II. FORMALISM

In this section we shall introduce basic notions about the RRPA (RQRPA) and RPA (QRPA) methods, as well as the definitions of the basis and generators of the algebra used to obtain exact solutions. Since most of the equations are well known, we are introducing them for the sake of completeness and long discussions on the formalism will be avoided. Each subsection of the present section will include the set of equations corresponding to a definite model situation and for it the exact solution will be shown together with the RPA (QRPA) and RRPA (RQRPA) solutions.

Monopole pairing excitations, monopole particle-hole excitations and quadrupole particle-hole excitations will be described in subsections A, B, and C, respectively. Each set of excitations is constructed by linear superpositions of like-

quasiparticle pairs in the superfluid phase or like-particle-hole pairs in the normal phase. In addition, two-particle correlations will be treated to describe pairing vibrations.

A. Monopole pair excitations

The correlated states of a system of N particles moving in two levels, each with a degeneracy 2Ω, are described by the Hamiltonian

$$H=\frac{\epsilon}{2}(N_2-N_1)-G\Omega(A_2^\dagger+A_1^\dagger)(A_2+A_1), \tag{1}$$

where

$$A_\sigma^\dagger=\frac{1}{\sqrt{\Omega}}\sum_{m>0} a_{\sigma m}^\dagger a_{\sigma m}^\dagger,$$

$$A_\sigma=(A_\sigma^\dagger)^\dagger, \tag{2}$$

$$N_\sigma=\sum_m a_{\sigma m}^\dagger a_{\sigma m}.$$

ϵ is the energy spacing between levels and the operator $a_{\sigma m}^\dagger$ ($a_{\sigma m}$) creates (annihilates) one-particle states denoted by $\{\sigma,m\}$, where $\sigma=1,2$ is the index associated with the upper (2) and lower (1) levels and $m=1,\ldots 2\Omega$ reads for the substates of each shell.

The operators $A_\sigma^\dagger, A_\sigma$ and N_σ obey the SU(2) algebra. Exact eigenvalues and eigenvectors are obtained by a diagonalization in the basis

$$|k,n-k\rangle=\sqrt{\frac{(\Omega-k)!(\Omega-n+k)!}{k!\Omega!(n-k)!\Omega!}}A_1^{\dagger k}A_2^{\dagger(n-k)}|\phi\rangle, \tag{3}$$

where $A_1|\phi\rangle=0$, $0\leqslant k\leqslant n$ and $2n=N$ is the number of fermions. The nonvanishing matrix elements of H are written

$$\begin{aligned}\langle k',n-k'|H|k,n-k\rangle &= \epsilon(n-2k)\delta_{k',k}-G[k(\Omega-k+1)\\ &\quad+(n-k)(\Omega-n+k+1)]\delta_{k',k}\\ &\quad-G\sqrt{(k+1)(n-k)(\Omega-k)(\Omega-n+k+1)}\,\delta_{k',k+1}\\ &\quad-G\sqrt{k(n-k+1)(\Omega-k+1)(\Omega-n+k)}\,\delta_{k',k-1}.\end{aligned} \tag{4}$$

1. The RPA approximation for monopole pairing vibrations

The solutions given by the random-phase approximation (RPA), for the spectrum of H in the normal phase, can be written in terms of addition (a) and removal (r) one-phonon operators. For brevity, we shall introduce the equations for the addition modes (similar equations are obtained for the removal mode by replacing particles by holes) [25]. Thus

$$\Gamma_a^\dagger=X_aA_2^\dagger-Y_aA_1^\dagger. \tag{5}$$

The harmonic version of H in the phonon basis is obtained from the commutator

$$[H,\Gamma_a^\dagger]=\omega_a\Gamma_a^\dagger, \tag{6}$$

with the eigenvalue $\omega_a=\epsilon\sqrt{1-\Omega G/(\epsilon/2)}$.

The forward (X) and backward (Y) going amplitudes are written [26]

$$X_a=\frac{\Lambda_a\sqrt{\Omega}}{\epsilon-\omega_a},\quad Y_a=\frac{\Lambda_a\sqrt{\Omega}}{\epsilon+\omega_a}, \tag{7}$$

with $\Lambda_a=G\sqrt{\Omega}/\sqrt{\omega_a/\epsilon}$.

2. The RPA approximation in the superfluid phase (QRPA)

After applying the BCS transformation [3] the Hamiltonian H, Eq. (1), is written as

$$\begin{aligned}H=&\sum_j E_jN_j-\frac{\Omega G}{2}\sum_{ij}(u_i^2u_j^2+v_i^2v_j^2)(A_i^\dagger A_j+A_j^\dagger A_i)\\ &+\frac{\Omega G}{2}\sum_{ij}(u_i^2v_j^2+u_j^2v_i^2)(A_i^\dagger A_j^\dagger+A_jA_i)\\ &+\sqrt{\Omega}G\sum_{ij}u_iv_i(u_j^2-v_j^2)(A_j^\dagger N_i+N_iA_j)\\ &-G\sum_{ij}u_iv_iu_jv_jN_iN_j.\end{aligned} \tag{8}$$

The quasiparticle energies ($E_1=E_2=E$) and the pairing gap parameter (Δ) are given by $E=\Omega G$ and $\Delta=\Omega G\sqrt{1-[(\epsilon/2)/\Omega G]^2}$, respectively. The BCS occupation factors are defined by $u_1=v_2=u=1/\sqrt{2}\sqrt{1-(\epsilon/2)/E}$ and $u_2=v_1=v=1/\sqrt{2}\sqrt{1+(\epsilon/2)/E}$, while the operators $A_\sigma^\dagger$, A_σ and N_σ of Eq. (8) are the same as Eq. (2) but written in terms of quasiparticle creation and annihilation operators. We have assumed $N=2\Omega$, and the Fermi level $\lambda_F=0$. The QRPA phonon operator is a linear superposition of two-quasiparticle states with the eigenvalue $\omega=2\Delta$. These equations are correct at leading order in the parameter Ω. Self-insertions in the unperturbed quasiparticle term are neglected as well as the self-energy corrections to quasiparticle and phonon energies due to the quasiparticle-phonon coupling. These contributions are of lower order in Ω and for the schematic Hamiltonian (1) the omission of these corrections will not affect the QRPA or the RQRPA results in a significant manner. The use of these corrections and the use of adjusted single-particle energies around the Fermi energy in the RQRPA and RRPA equations have been considered in [20]. We shall discuss them in Sec. III.

3. The renormalized RPA

The effect due to ground-state correlations, upon the RPA (and QRPA) amplitudes and eigenvalues was studied by Hara [11] and applied to the RPA treatment of monopole excitations by Catara *et al.* [12]. By scaling the pair creation operators $A^\dagger$, in the normal phase,

$$\tilde{A}_\sigma^\dagger=D_\sigma^{-1/2}A_\sigma^\dagger, \tag{9}$$

the Hamiltonian (1) is written

$$H=\frac{\epsilon}{2}(N_2-N_1)$$
$$-G\Omega(\sqrt{D_2}\tilde{A}_2^{+}+\sqrt{D_1}\tilde{A}_1^{+})(\sqrt{D_2}\tilde{A}_2+\sqrt{D_1}\tilde{A}_1). \tag{10}$$

The factors D should be determined self-consistently. The renormalized RPA equations, which define the scaled phonon creation operator

$$\tilde{\Gamma}_a^{\dagger}=\tilde{X}_a\tilde{A}_2^{\dagger}-\tilde{Y}_a\tilde{A}_1^{\dagger}, \tag{11}$$

(and similarly for the removal one-phonon operator) are written

$$\tilde{X}_a^2-\tilde{Y}_a^2=1,$$
$$(\epsilon-\Omega GD_2-\tilde{\omega}_a)\tilde{X}_a-\Omega G\sqrt{D_1D_2}\tilde{Y}_a=0, \tag{12}$$
$$-\Omega G\sqrt{D_1D_2}\tilde{X}_a-(\epsilon-\Omega GD_1+\tilde{\omega}_a)\tilde{Y}_a=0,$$

with

$$D_1=\frac{1}{1+\tilde{Y}_a^2},\quad D_2=\frac{1}{1+\tilde{Y}_r^2}.$$

It implies that self-consistency can only be achieved by treating both the addition and removal modes simultaneously. The above structure, for the coefficients D, does not include an additional dependence on single-particle occupation numbers [21,22,24]. These occupation numbers are calculated in the correlated ground state and they appear in the definition of the RPA (QRPA) metric [22,24]. In the approach developed in [22] and [24] the right-hand side of the QRPA (RPA) matrix equations [2] includes a diagonal matrix with matrix elements defined by the ground-state expectation values of the number operator, for each single-particle level. In this form, the local variation of the central single-particle density produced by phonon excitations is the source of RPA density fluctuations [3]. However, and in the contexts of the present approximations, these corrections are also of lower order, as compared to the RPA (QRPA) leading-order terms [23].

4. The renormalized QRPA approximation

In this approximation the phonon operator reads

$$\tilde{\Gamma}^{\dagger}=\tilde{X}(\tilde{A}_2^{\dagger}-\tilde{A}_1^{\dagger})-\tilde{Y}(\tilde{A}_2-\tilde{A}_1). \tag{13}$$

After the renormalization given in Eq. (9) is applied we have obtained the following equations:

$$2(\tilde{X}^2-\tilde{Y}^2)=1,$$
$$\left(2E-\frac{\epsilon^2}{E}D-\tilde{\omega}\right)\tilde{X}-\frac{\epsilon^2}{E}D\tilde{Y}=0, \tag{14}$$
$$-\frac{\epsilon^2}{E}D\tilde{X}-\left(2E-\frac{\epsilon^2}{E}D+\tilde{\omega}\right)\tilde{Y}=0$$

with

$$D=\frac{1}{1+\tilde{Y}^2}. \tag{15}$$

As we have said above, these definitions are valid at leading order in Ω. Further corrections, to the single-quasiparticle occupation factors or to the quasiparticle and phonon energies, produce the mixing of orders. The structure of the renormalized equations including these corrections are given, for instance, in [24].

5. The Dyson boson mapping

For the adopted single-particle basis, the fermionic operators $N_\sigma, A_\sigma^{\dagger}, A_\sigma$ can be mapped onto a bosonic $(b_\sigma^{\dagger}, b_\sigma)$ space which preserves the original SU(2) algebra, thus [17]

$$\sqrt{\Omega}A_2^{\dagger}=b_2^{\dagger}(\Omega-b_2^{\dagger}b_2)(=\tilde{b}_2^{\dagger}),$$
$$\sqrt{\Omega}A_2=b_2,$$
$$N_2=b_2^{\dagger}b_2, \tag{16}$$
$$\sqrt{\Omega}\overline{A_1^{\dagger}}=b_1^{\dagger}(\Omega-b_1^{\dagger}b_1)(=\tilde{b}_1^{\dagger}),$$
$$\sqrt{\Omega}\overline{A_1}=b_1,$$
$$N_1=\Omega-b_1^{\dagger}b_1,$$

with $[b_j,b_i^{\dagger}]=\delta_{ij}$, and $\overline{A_1^{\dagger}}=A_1$.

In this space, bra and ket vectors are given by

$$|k,n-k\rangle=\sqrt{\frac{(\Omega-k)!}{k!\Omega!}\frac{(\Omega-n+k)!}{(n-k)!\Omega!}}\tilde{b}_2^{\dagger(n-k)}\tilde{b}_1^{\dagger(\Omega-k)}|0\rangle,$$
$$\langle nk|=\sqrt{\frac{(\Omega-k)!}{k!\Omega!}\frac{(\Omega-n+k)!}{(n-k)!\Omega!}}\langle 0|b_2^{(n-k)}b_1^{(\Omega-k)} \tag{17}$$

with $0\leqslant k\leqslant n$. At leading order in powers of Ω, the matrix elements of the Hamiltonian are written

$$\langle k',n-k'|H|k,n-k\rangle$$
$$=\{2\epsilon(n-2k)-G[k(\Omega-k+1)$$
$$+(n-k)(\Omega-n+k+1)]\}\delta_{k',k}$$
$$-G\Omega\sqrt{(k+1)(n-k)}\left(1-\frac{k}{2\Omega}\right)$$
$$\times\left(1-\frac{n-k-1}{2\Omega}\right)\delta_{k',k+1}-G\Omega\sqrt{k(n-k+1)}$$
$$\times\left(1-\frac{k-1}{2\Omega}\right)\left(1-\frac{n-k}{2\Omega}\right)\delta_{k',k-1}. \tag{18}$$

B. Monopole particle-hole excitations

The Hamiltonian of the Lipkin-Meschkov-Glick (LMG) model [27]

$$H=\epsilon K_0-\frac{1}{2}V(K_+^2+K_-^2) \tag{19}$$

describes monopole particle-hole excitations in a two-level single-particle space, where

$$K_+=\sum_m a_{2m}^\dagger a_{1m},$$

$$K_-=K_+^\dagger,$$

$$K_0=\frac{1}{2}\sum_m (a_{2m}^\dagger a_{2m}-a_{1m}^\dagger a_{1m}).$$

The exact solution of the model is obtained by considering the set of vectors

$$|k\rangle=\sqrt{\frac{(2\Omega-k)!}{k!(2\Omega)!}}K_+^k|\phi\rangle, \quad 0\leqslant k\leqslant 2\Omega, \tag{20}$$

$|\phi\rangle$ is a pure fermion state obeying $K_-|\phi\rangle=0$. The nonvanishing matrix elements of H are

$$\begin{aligned}\langle k'|H|k\rangle=&\epsilon(k-\Omega)\delta_{k',k}-\frac{1}{2}V\\&\times\sqrt{(k+1)(k+2)(2\Omega-k)(2\Omega-k-1)}\,\delta_{k',k+2}\\&-\frac{1}{2}V\sqrt{k(k-1)(2\Omega-k+1)(2\Omega-k+2)}\,\delta_{k',k-2}.\end{aligned} \tag{21}$$

1. The RPA approximation in the LMG model

The one-phonon creation operator is defined by

$$\Gamma^\dagger=\sqrt{2\Omega}(XK_+-YK_-), \tag{22}$$

and the corresponding RPA equation of motion yields the eigenfrequency $\omega=\epsilon\sqrt{1-(2\Omega V/\epsilon)^2}$.

2. The renormalization of monopole particle-hole excitations

Introducing the scaling $\tilde{K}_+=D^{-1/2}K_+$, the Hamiltonian [Eq. (19)] can be written

$$H=\epsilon K_0-\frac{1}{2}VD(\tilde{K}_+^2+\tilde{K}_-^2) \tag{23}$$

with the corresponding renormalized one-phonon operator

$$\tilde{\Gamma}^\dagger=\sqrt{2\Omega}(\tilde{X}\tilde{K}_+-\tilde{Y}\tilde{K}_-). \tag{24}$$

The RPA equation of motion leads to the system of equations

$$\begin{aligned}&2\Omega(\tilde{X}^2-\tilde{Y}^2)=1,\\&(\epsilon-\tilde{\omega})\tilde{X}-2\Omega VD\tilde{Y}=0,\\&-2\Omega VD\tilde{X}-(\epsilon+\tilde{\omega})\tilde{Y}=0,\end{aligned} \tag{25}$$

where $D=1/(1+2\tilde{Y}^2)$.

Also here we have omitted the inclusion of monopole corrections to the single-particle energies, as well as additional corrections to the D factors given by higher powers of $\tilde{Y}$ in Eq. (25).

3. Boson mapping of the LMG Hamiltonian

We can transform the fermionic operators $K_0,K_\pm$ onto a boson $(b^\dagger,b)$ space which preserves the original SU(2) algebra [17]

$$\begin{aligned}&K_+=b^\dagger(2\Omega-b^\dagger b)(=\tilde{b}^\dagger),\\&K_-=b,\\&K_0=b^\dagger b-\Omega\end{aligned} \tag{26}$$

with $[b,b^\dagger]=1$.

In this space the bra and ket vectors are given by

$$\begin{aligned}&|k\rangle=\sqrt{\frac{(2\Omega-k)!}{k!(2\Omega)!}}\tilde{b}^{\dagger k}|0\rangle,\\&\langle k|=\sqrt{\frac{(2\Omega-k)!}{k!(2\Omega)!}}\langle 0|b^k\end{aligned} \tag{27}$$

with $0\leqslant k\leqslant 2\Omega$. The nonvanishing matrix elements of H are

$$\begin{aligned}\langle k'|H|k\rangle=&\epsilon k\delta_{k',k}-\frac{1}{2}V\\&\times\sqrt{(k+1)(k+2)(2\Omega-k)(2\Omega-k-1)}\,\delta_{k',k+2}\\&-\frac{1}{2}V\sqrt{k(k-1)(2\Omega-k+1)(2\Omega-k+2)}\,\delta_{k',k-2}.\end{aligned} \tag{28}$$

This result is the same as the one obtained in the exact treatment of the model [27].

To leading order in Ω, the matrix elements of the Hamiltonian are given by

$$\begin{aligned}\langle k'|H|k\rangle\cong&\epsilon k\delta_{k',k}-V\Omega\sqrt{(k+1)(k+2)}\,\delta_{k',k+2}\\&-V\Omega\sqrt{k(k-1)}\,\delta_{k',k-2}.\end{aligned} \tag{29}$$

C. Quadrupole particle-hole excitations

To complete the study of the approximations described in the previous subsections we shall introduced the separable quadrupole-quadrupole interaction [4,5]

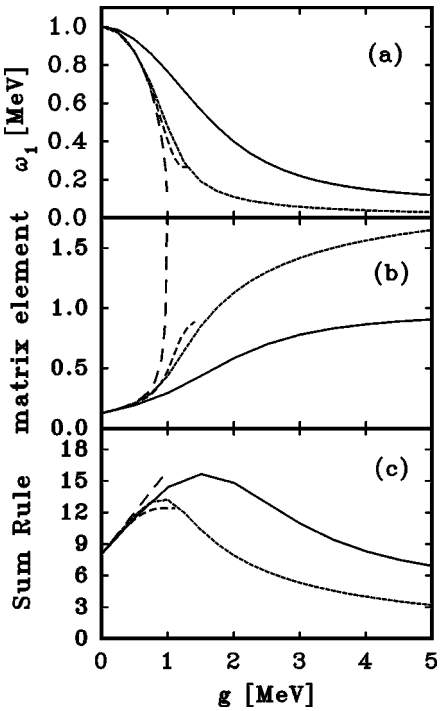

FIG. 1. Results corresponding to monopole particle-hole excitations in different approximations as a function of $g=2\Omega V$. The excitation energy of the first excited state (ω_1) is shown in inset (a), the matrix elements of a monopole particle-hole operator [12] which induces transitions between the ground state and the first excited state are shown in (b), and the contributions to the monopole energy weighted sum-rule are shown in inset (c). The solid curve represents both the exact result and the result of Dyson's boson mapping to all order; the long-dashed line represents the RPA (QRPA) result; the short-dashed line shows the result based in Dyson's boson mapping at leading order in Ω, the results of the RRPA (RQRPA) are shown by the dotted line.

$$H=H_0+H_{\rm int},$$

$$H_0=\sum_j E_jN_j, \tag{30}$$

$$H_{\rm int}=-g_2\sum_\mu P^\dagger_{2\mu}P_{2\mu}+\kappa_2\sum_\mu Q^\dagger_{2\mu}Q_{2\mu},$$

where N is the quasiparticle-number operator and

$$P^\dagger_{2\mu}=\sum_{j_1,j_2} p_2(j_1,j_2)(u_{j_1}u_{j_2}A^\dagger_{2\mu}-v_{j_1}v_{j_2}A_{\overline{2\mu}}),$$

$$Q^\dagger_{2\mu}=\sum_{j_1,j_2} t_2(j_1,j_2)(A^\dagger_{2\mu}+A_{\overline{2\mu}}), \tag{31}$$

are two-quasiparticle components of the particle-particle and particle-hole quadrupole operator. The notation is given in [28]. It should be noted that quasiparticle states are taken as a crude representation of the mean field and that self-consistent corrections to the single particle (or single-quasiparticle) energies originated in the vacuum expectation value of the quadrupole operator are not considered. The lack of self-consistency introduced in this fashion will not affect the main trend of the renormalized results, as compared to the exact solutions. Like in the previous subsections the present discussion is restricted to the analysis of leading-order effects.

Since an exact solution of the pairing plus quadrupole Hamiltonian in more than one shell cannot be formulated in terms of generators of a given algebra we shall restrict the analysis of the solutions to the one-shell limit. For this case (N active particles outside a core)

$$H_{\rm int}=H_{22}+H_{40}, \tag{32}$$

H_{22} and H_{40} are the two and four quasiparticle terms of the Hamiltonian. Realistic values of the corresponding effective coupling constants g_{22} and g_{40}, defined by

$$g_{22}=\kappa_2|t_2|^2-\frac{1}{2}g_2|p_2|^2(u^4+v^4),$$

$$g_{40}=\kappa_2|t_2|^2+g_2|p_2|^2u^2v^2, \tag{33}$$

are obtained by fixing $g_{22}=0$, and, consequently,

$$\kappa_2=\frac{g_2}{2R_0^4}\frac{u^4+v^4}{u^2v^2}. \tag{34}$$

With this set of parameters the Hamiltonian (30) reduces to the familiar form [27]

$$H=\sum_j E_jN_j+g_{40}\frac{1}{2}\sum_\mu (A^\dagger_{2\mu}A^\dagger_{\overline{2\mu}}+A_{\overline{2\mu}}A_{2\mu}), \tag{35}$$

which becomes linear in the quadrupole-phonon basis, by applying the QRPA transformations. The solution for the corresponding QRPA eigenvalue is written $\omega=2E\sqrt{1-(g_{40}/E)^2}$.

1. RQRPA of the quadrupole-quadrupole interaction

Introducing the scaling (9) at the level of the two-quasiparticle operator $A^\dagger_{2\mu}$ the Hamiltonian of Eq. (35) is written

$$H=EN+g_{40}\frac{1}{2}\sum_\mu D^{1/2}_\mu(\tilde{A}^\dagger_{2\mu}\tilde{A}^\dagger_{\overline{2\mu}}+\tilde{A}_{\overline{2\mu}}\tilde{A}_{2\mu})D^{1/2}_\mu. \tag{36}$$

Thus, the RQRPA leads to the system of equations

$$\tilde{X}^2_{\mu n}-\tilde{Y}^2_{\mu n}=1,$$

$$(2E-\tilde{\omega}_n)\tilde{X}_{\mu n}-2g_{40}\tilde{D}_\mu\tilde{Y}_{\mu n}=0, \tag{37}$$

$$2g_{40}\tilde{D}_\mu\tilde{X}_{\mu n}-(2E+\tilde{\omega}_n)\tilde{Y}_{\mu n}=0$$

for the renormalized amplitudes. The validity of these equations is restricted by the same considerations following Eq. (12), of Sec. II A.

2. Dyson boson mapping of the quadrupole Hamiltonian

It is performed by introducing the boson mapping

$$A_{2\mu}^{\dagger}=B_{2\mu}^{\dagger}-\sum_{\rho M_\rho \nu M_\nu \sigma M_\sigma}\Gamma_{2\mu\sigma M_\sigma}^{\rho M_\rho \nu M_\nu}B_{\rho M_\rho}^{\dagger}B_{\nu M_\nu}^{\dagger}B_{\sigma M_\sigma}(=\tilde{B}_{2\mu}^{\dagger}),$$

$$A_{2\mu}=B_{2\mu}, \tag{38}$$

$$a_{\mu\nu}=\sum_{\rho M_\rho \sigma M_\sigma}\Gamma_{2\mu\sigma M_\sigma}^{2\mu\rho M_\rho}B_{\rho M_\rho}^{\dagger}B_{\sigma M_\sigma}$$

with $[B_{\rho M_\rho},B_{\sigma M_\sigma}^{\dagger}]=\delta_{\rho\sigma}\delta_{M_\rho M_\sigma}$.

In the present case the underlying algebra is given by the SO(2Ω) [29] representation defined by the commutators

$$[A_{2\mu},A_{2\nu}^{\dagger}]=\delta_{\mu\nu}-2a_{\mu\nu},$$

$$[N,A_{2\mu}^{\dagger}]=2A_{2\mu}^{\dagger}, \tag{39}$$

$$[N,A_{2\mu}]=-2A_{2\mu}.$$

Vectors in this space are defined by

$$|k(\mu)\rangle=N(k(\mu))\tilde{B}_{2\mu}^{\dagger k(\mu)}|0\rangle,$$

$$\langle k(\mu)|=N(k(\mu))\langle 0|B_{2\mu}^{k(\mu)}, \tag{40}$$

the normalization factor $N(k(\mu))$ reads

$$N(k(\mu))=\frac{1}{\sqrt{k(\mu)!\Omega[\Omega-f(\mu)]\cdots[\Omega-(k(\mu)-1)f(\mu)]}} \tag{41}$$

with $f(\mu)=\frac{1}{2}\Gamma_{2\mu 2\mu}^{2\mu 2\mu}$, and $0\leqslant k\leqslant\Omega$ $[k=\Sigma_\mu k(\mu)]$.

The nonvanishing matrix elements of H are

$$\begin{aligned}\langle k'|H|k\rangle=&\,2Ek\delta_{k',k}\\&-g_{40}\sum_\mu\sqrt{[k(\mu)+1][k(-\mu)+1]}\sqrt{[\Omega-k(\mu)f(\mu)][\Omega-k(-\mu)f(-\mu)]}\,\delta_{k'(\mu),k(\mu)+1}\delta_{k'(-\mu),k(-\mu)+1}\\&-g_{40}\sum_\mu\sqrt{k(\mu)k(-\mu)}\sqrt{[\Omega-(k(\mu)-1)f(\mu)][\Omega-(k(-\mu)-1)f(-\mu)]}\,\delta_{k'(\mu),k(\mu)-1}\delta_{k'(-\mu),k(-\mu)-1}.\end{aligned} \tag{42}$$

III. RESULTS AND DISCUSSION

In this section we shall present the results of the calculations, for the excitation energies and transition matrix elements of relevant operators, performed within the different approximations introduced in the above section.

The exact results, for the case of monopole pairing vibrations, have been obtained by adopting the parametrization $\Omega=20$, $N=40$ and $\epsilon=1$, for the shell degeneracy, the number of particles and the energy spacing between levels, respectively. For the case of monopole particle-hole excitations, the adopted values are $\Omega=4$, $N=4$ and $\epsilon=1$. The conclusions extracted from the present results will be limited by the schematic nature of the interactions which we have used to calculate RQRPA (RRPA) quantities. On the other hand, the results have been consistently obtained at a given order in the expansion parameter Ω. Before presenting our results we would like to comment, briefly, on some of the existing results of the RRPA (RQRPA) [19–24]. Let us concentrate first on the effects which we have consistently neglected in our calculations, like self-insertions and self-energy corrections to single-particle and phonon energies, and the self-consistent coupling between the mean field and the residual interactions. It is indeed true that the strength of the residual interaction should be fixed in a consistent manner. It will depend always on the single-particle basis and the renormalization of the couplings will also be a function of the number of single-particle states and their energies. In realistic situations single-particle properties, i.e., the sequence and density of levels around the Fermi surface, pairing properties, and vibrational properties in double-even- and in double-odd-mass nuclei, depend upon each other. In principle, for a given two-body interaction and in a given single-particle basis, the HF mean field can be solved and the corresponding residual interactions can be treated consistently. In other approximations, the mean field is represented by empirical single-particle or quasiparticle levels taken from odd-even-mass and even-odd-mass nuclei and their mass differences and the strength of the residual interaction is fixed by reproducing the energy of the first excited state of the corresponding even-even- or odd-odd-mass nuclei [20]. This procedure leads to nonconsistent couplings. In this respect, the removal of the spurious dipole state by selecting a suitable strength at the RPA level of approximation is a good example [3]. For schematic Hamiltonians the contributions of single particle and phonon degrees of freedom can be ordered in powers of the shell degeneracy Ω. In this respect the RPA is a theory of small-amplitude vibrations around the

minima given by the single-particle mean field [3]. Both the single-particle or quasiparticle energies and the RPA (QRPA) energies are of order Ω and the couplings are of order $1/\sqrt{\Omega}$. Self-insertions in the quasiparticle energies (i.e., gv^4) and the self-energy corrections due to the coupling to phonon states are of lower order [3]. For large values of Ω these corrections can be neglected. Of course, for realistic situations they can affect the energy spacing between states above and below the Fermi surface. For the cases which we have discussed, we have consistently worked at leading order in Ω to avoid the additional complications of mixing orders and overcompleteness. The inclusion of self-insertions and self-energy corrections, both to the fermions and phonons, will certainly change the point where the RPA collapses and the point where the renormalized RPA shows a departure in respect to it. In schematic situations, like the situations discussed in the text, the overall trend remains unaffected [22]. From the published evidence we see that this kind of procedure, which for schematic models does not reproduce the exact solution either, does not work beyond the RPA-phase transition point, as shown by Delion *et al.* [19].

The results shown in Fig. 1, which correspond to monopole particle-hole excitation in the LMG model, do indeed reproduce the results of Catara *et al.* [12] concerning the value of the energy of the first excited state. This figure shows the characteristic collapse of the RPA eigenvalue at a certain critical value of the coupling constant $g=2\Omega V$. By using the renormalized RPA method the collapse (i.e., $w \rightarrow 0$) is avoided, a trend which is also shown by the exact solution. The solution obtained by using Dyson's boson mapping at leading order is indeed very similar to the RRPA solution. However they are still very different from the exact solution, for values of g larger than the critical value ($g_c=1$) of Fig. 1. From these results one may be tempted to conclude that the RRPA method works fairly well, in spite of the fact that the RPA, RRPA and exact wave functions of the first excited state look very different, as can be seen from the curves of Fig. 1(c). One interesting feature of these curves is the fact that the Dyson boson mapping method at leading order and the renormalized RPA yield comparable results around g_c but both approximations differ strongly from the exact solution.

The same situation can be illustrated by the overlaps between the wave functions obtained in the exact and RRPA calculations. As shown in Table I, the RRPA and exact wave functions look very different passing by the critical value g_c.

The behavior of the approximations in the vicinity of a phase transition is better illustrated, perhaps, by the case of monopole pairing vibrations. The well-known separation between the normal and superfluid phases, as a function of the pairing coupling constant, is exhibited by the results for the energy of the first excited state, as shown in Fig. 2(a). The curves look very familiar [18] and the discrepancy shown by the renormalized RPA(QRPA), as compared with the exact solutions, is evident. While the RPA(QRPA) produces a zero-energy eigenvalue at $g_c=0.5$ (for this case $g=\Omega G$) the results of the renormalized approximations cross the critical point without vanishing. In fact, the results of Dyson-boson mapping are better than other approximations, as compared with the exact results. Again in this case the strong differences between exact and renormalized wave functions at

TABLE I. Overlaps between RPA and RRPA wave functions. (a) Overlaps between exact and RPA (second column) and RRPA wave functions (third column) for monopole particle-hole excitations. The values of the coupling constant $g=2\Omega V$ are given in the first column. (b) Overlaps between exact and RPA (second column) and RRPA wave functions (third column) for monopole pairing excitations in the normal phase. The values of the coupling constant $g=\Omega G$ are given in the first column. (c) Overlaps corresponding to monopole pairing excitations in the superfluid phase. (d) Overlaps corresponding to quadrupole excitations, with $g=g_{40}\Omega$.

g	RPA	RRPA
	(a)	
0.0	1.00000	1.00000
0.2	1.00023	0.99764
0.4	1.00298	0.99143
0.6	1.01674	0.98403
0.8	1.07399	0.97999
1.0		0.98238
2.0		0.82105
3.0		0.62544
4.0		0.50384
4.8		0.43896
	(b)	
0.0	1.00000	1.00000
0.2	0.99369	0.99335
0.4	0.97218	0.94512
0.6		0.62930
0.8		0.17723
1.0		0.04954
1.2		0.01833
1.4		0.00853
1.6		0.00469
1.8		0.00291
2.0		0.00197
	(c)	
0.20000		0.00000
0.40000		0.00001
0.60000	0.00484	0.00279
0.80000	0.08182	0.04885
1.00000	0.28032	0.16962
1.20000	0.51021	0.31109
1.40000	0.69857	0.42795
1.60000	0.82750	0.50845
1.80000	0.90481	0.55705
2.00000	0.94398	0.58190
	(d)	
0.00000	1.00000	1.00000
0.20000	0.97168	0.92420
0.40000	0.89254	0.73287
0.50000	0.86085	0.62202
0.60000	0.99777	0.52002
0.80000		0.36804
1.00000		0.27603
1.20000		0.21940
1.40000		0.18233
1.60000		0.15657
1.80000		0.13777

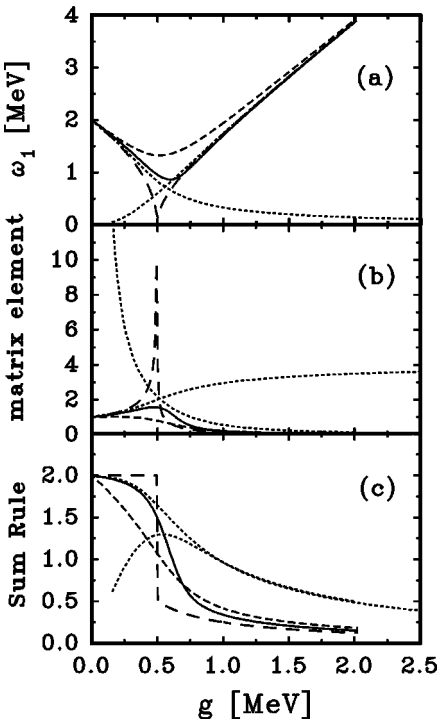

FIG. 2. Monopole pairing vibrations. Energy, matrix element of the two-particle transfer operator and contribution to the energy weighted sum rule, for transitions connecting the ground state and the first excited one-phonon state. (a) shows the excitation energy of the first excited state given by the different approximations, as a function of the coupling constant $g=\Omega G$; (b) shows the square matrix element of the two-particle transfer operator corresponding to the transition from the ground state to the first excited state; and (c) shows the contribution to the energy weighted sum rule for the same transition. The convention used to display the results is given in the caption to Fig. 1.

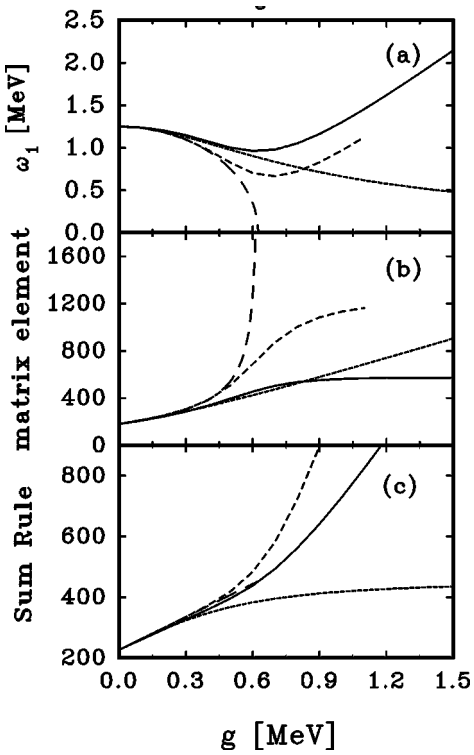

FIG. 3. Results corresponding to quadrupole excitations in different approximations and for different values of $g=\Omega g_{40}$. The excitation energies are shown in inset (a), the transition-matrix element of the quadrupole operator, for transitions between the ground state and the first excited state, is shown in (b), and its contributions to the energy weighted sum-rule are shown in inset (c). The curves are denoted as described in the caption to Fig. 1.

each side of the phase-transition point are reflecting upon the dependence of the contributions to the sum rule (for transitions induced by the two-particle transfer operator) [Fig. 2(c)].

Finally, the results corresponding to quadrupole excitations in a single shell, for the various methods discussed in the text, are shown in Fig. 3. The features of the solutions, for this case, are very much the same as those of the previous cases. The fact that the renormalized wave functions and the exact ones are different is shown clearly by the curves of Fig. 3(c).

IV. SUMMARY AND OUTLOOK

The renormalization technique of Hara [11], as formulated by Catara *et al.* [12], was used to treat two-particle and particle-hole multipole excitations in schematic models. Although the eigenvalues obtained with the RRPA (RQRPA) are similar to the eigenvalues given by the exact and standard RPA(QRPA) methods, the renormalization procedure seemingly fails in reproducing the wave functions. The strong departure from the exact wave functions across and passing by the point where the RPA (QRPA) collapses, suggests that the renormalization method may not be able to account correctly for the correlations induced by the Hamiltonian. Obviously, these conclusions are limited by the very schematic structure of the models so far considered but it is worth mentioning that the differences between the standard and renormalized RPA and exact solutions, may depend upon each Hamiltonian. While the agreement between exact and renormalized results, for the eigenvalues, is good for some cases, like the case considered in Ref. [12], they are not so good for other cases, like for monopole pairing and quadrupole excitations. Particularly, the crossing of a phase-transition point, as shown in the case of pairing vibrational modes is a warning about the use of the renormalization technique in more realistic situations.

As seen from the results shown in this work it appears that the renormalization procedure seemingly works correctly in the regions where the naive RPA also works, that is in the region before the collapse, but it is also seen from the above results that a departure from the exact results is observed even in this region. As mentioned before, the difference between exact and renormalized methods may also depend on the particular Hamiltonian used in the calculations. These features may also suggest that in actual applications of the renormalized RPA approach to realistic cases the question about its validity nearby a transition point still deserves to be discussed.

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