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Physical representations of Gamow states in a rigged Hilbert space

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Abstract

Resonant Gamow States (GS) are constructed in a rigged Hilbert space (RHS) $(\xi, \mathcal{H}, \xi')$ starting from Dirac's formula. It is shown that the expectation value of a self-adjoint operator acting on a GS is real. The validity of recently proposed approximations to calculate expectation values on resonant states is discussed.

The treatment of the continuum and the inclusion of decaying states in the definition of the nuclear response is a long-standing problem [1–3]. The inclusion of resonant states in the one-body Green Function has been studied years ago by Tore Berggren [4,5]. More recently, the use of these states to calculate one-particle and collective excitations in finite nuclei has been proposed by Liotta et al. [6]. A key component of these microscopic descriptions of nuclear properties including the continuum, as shown in [6] and [7] is the use of the techniques developed by Vertse et al. [8].

Several methods have been developed in connection with the treatment of GS [9–11]. The equivalence between some of these methods and the correspondence between Bergreen's and Mittag-Lefler's representations have been explored in dealing with the use of GS in nuclear structure problems [6]. Mathematical properties of GS, in the framework of the Hamiltonian formalism, have been studied by Sudarshan and collaborators [12]. Bohm et al. [13,14] have shown that the RHS is a suitable framework to describe idealized resonances as generalized eigenvectors of a self-adjoint Hamiltonian with complex eigenvalues. The overlap between GS and wave packets of the Breit-Wigner form has been discussed by Romo [10] by using techniques of analytic continuation. However, difficulties associated with the interpretation of expectation values of operators on GS have prevented a more extended use of these states in nuclear structure calculations. The possibility of defining expectation values of operators in a resonant state has been studied by Tore Berggren in a recent work [15].

In this work we shall show that the results of [15] are valid at leading order in Γ, the imaginary part of the energy of a GS. At variance with the usually adopted mathematical formalism [9,14] we shall use the concepts of tempered ultradistributions and Gelfand's triplets [16]. In the following, only the aspects of the derivation which are relevant to validate Berggren's approximation will be shown. The neces-

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sary mathematical details of the formalism are given in [17].

The space of analytical functionals ξ' (tempered ultradistributions) is the minimal space whose Fourier anti-transform accommodates real exponential functions as distributions. This space is the dual of the space of analytical test functions ξ. Together with the Hilbert space $\mathcal{H}$ one can construct the RHS or Gelfand's triplet (GT) [16,17] $\xi \subset \mathcal{H} \subset \xi'$. In this RHS a linear and symmetric operator A acting on ξ, which admits a self-adjoint prolongation $\bar{A}$ acting on $\mathcal{H}$, has a complete set of eigen-functionals on ξ' with real generalized eigenvalues [16]. Let us introduce the GT $(\tilde{\xi}, \mathcal{H}, \tilde{\xi}')$ which is related to $(\xi, \mathcal{H}, \xi')$ by Fourier transforms. The Schwartz space $\mathcal{S}'$ of tempered distributions is included in ξ' and in $\tilde{\xi}'$. The extension to ξ' of Dirac's formula is given by [17,18]

$$\hat{\psi}_c(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1}{t-z} \hat{\psi}(t)\, dt\,, \tag{1}$$

where $\hat{\psi}(t) = \hat{\psi}_c(t+i0) - \hat{\psi}_c(t-i0)$.

Let us introduce a self-adjoint operator H, acting on $\mathcal{H}$, with the eigenstates (eigenvalues) given by $H|E_n\rangle = E_n|E_n\rangle$ (for $n \in \mathcal{N}$) and $H|E\rangle = E|E\rangle$ (for $E_0 < E < E_1$). Thus from Eq. (1) one can write

$$(\hat{\psi}(E_G))^* = \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{E_G{}^* - E} (\hat{\psi}(E))^*\, dE\,, \tag{2}$$

with $E_G = E_D + i\Gamma$, $\Gamma > 0$.

In Dirac's notation one has

$$(\hat{\psi}(E_G))^* = \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{E_G{}^* - E} \langle\psi|E\rangle\, dE\,. \tag{3}$$

We can now define

$$|E_G{}^*\rangle = \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{E_G{}^* - E} |E\rangle\, dE\,. \tag{4}$$

Then $\hat{\psi}(E_G) = \langle E_G|\psi\rangle$ and $(\hat{\psi}(E_G))^* = \langle\psi|E_G{}^*\rangle$. The state $|E_G{}^*\rangle$ is by definition a GS eigenstate of H, since

$$\begin{aligned}\langle\psi|H|E_G{}^*\rangle &= \frac{1}{2\pi i} \int_{E_0}^{E_1} \frac{1}{E_G{}^* - E} E(\psi(E))^*\, dE \\ &= E_G{}^*\langle\psi|E_G{}^*\rangle\,.\end{aligned} \tag{5}$$

The states $|E_G{}^*\rangle$ are normalizable and the norm is given by

$$\langle E_G|E_G^*\rangle = \frac{1}{4\pi^2\Gamma} \times \left[\arctan\left(\frac{E_1 - E_D}{\Gamma}\right) - \arctan\left(\frac{E_0 - E_D}{\Gamma}\right)\right]. \tag{6}$$

The diagonal matrix element of H between GS is given by the expression

$$\langle E_G|H|E_G^*\rangle = E_D + \frac{\Gamma}{2} \frac{\ln\left[\frac{(E_1-E_D)^2+\Gamma^2}{(E_0-E_D)^2+\Gamma^2}\right]}{\left[\arctan\left(\frac{E_1-E_D}{\Gamma}\right) - \arctan\left(\frac{E_0-E_D}{\Gamma}\right)\right]}\,. \tag{7}$$

Thus the imaginary part of this diagonal matrix element vanishes and for the limits $E_0 \to -\infty$ and $E_1 \to +\infty$ it gives $\langle E_G|H|E_G^*\rangle = E_D$.

The probability distribution associated to a GS is given by

$$P(E) = |\langle E|E_G^*\rangle|^2 = \frac{\Gamma}{(E-E_D)^2 + \Gamma^2} \times \frac{1}{\left[\arctan\left(\frac{E_1-E_D}{\Gamma}\right) - \arctan\left(\frac{E_0-E_D}{\Gamma}\right)\right]}\,. \tag{8}$$

In the limit $E_1 \to +\infty$, $E_0 \to -\infty$ the above equation yields the Breit-Wigner form proposed in [10,13].

Let us introduce the self-adjoint operator A, which is acting on the complete abstract space $\mathcal{H}_a$ [17],

$$A = \int_{-\infty}^{+\infty} |\lambda > \lambda\, d\sigma_a(\lambda) < \lambda|\,, \tag{9}$$

where $\sigma_a(\lambda)$ is given by

$$\sigma_a(\lambda) = \begin{cases} \sum_{n=-\infty}^{+\infty} \Theta(\lambda - \lambda_n) & \lambda < \lambda_\infty \\ \lambda & \lambda_\infty < \lambda < \lambda_a \end{cases} \tag{10}$$

and where Θ is a Heaviside step function. The expectation value of A between GS

$$<E_G|A|E_G{}^*> = \int_{-\infty}^{+\infty} <E_G|\lambda> \lambda \, d\sigma_a(\lambda) <\lambda|E_G{}^*>, \quad (11)$$

is real since $<E_G|\lambda> = (<\lambda|E_G^*>)^*$.

In the following we shall discuss the validity of the approximation proposed in [15] to calculate the expectation value of an operator in a resonant state. Following Berggren's notation, let us introduce the state $|k, \hat{k}, l>$ and the continuum wave function $<\boldsymbol{x}|k, \hat{k}, l> = \phi_k^{(+)}(\boldsymbol{r})$. Since the energy E is given by $E(\boldsymbol{k}) = k^2/2m$ the GS state can be defined by

$$|E_G{}^*> = \frac{\sqrt{\Gamma}}{i\sqrt{\pi/2}} \int_0^{+\infty} \frac{|E(\boldsymbol{k}), \hat{k}, l>}{E_G{}^* - E} dE, \quad (12)$$

and using the impulse representation it is written as

$$|E_G{}^*> = \frac{\sqrt{\Gamma}}{i\sqrt{\pi/2}} \int_0^{+\infty} \sqrt{\frac{k}{m}} \frac{|k, \hat{k}, l>}{E_G{}^* - E(\boldsymbol{k})} dk. \quad (13)$$

Consequently, for the expectation value of A one has the expression

$$<E_G|A|E_G{}^*> = \frac{2\Gamma}{\pi} \sum_{l,l'} \int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m} \times \frac{<k', \hat{k}', l'|A|k, \hat{k}, l>}{(E(\boldsymbol{k}') - E_G)(E(\boldsymbol{k}) - E_G{}^*)}. \quad (14)$$

We are now in conditions to compare the result provided by the present method and Berggren's conjecture, namely: $<A> = \mathrm{Re}<E_G{}^*|A|E_G{}^*>$, where

$$<E_G{}^*|A|E_G{}^*> = \frac{2\Gamma}{\pi} \sum_{l,l'} \int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m} \times \frac{<k', \hat{k}', l'|A|k, \hat{k}, l>}{(E(\boldsymbol{k}') - E_G{}^*)(E(\boldsymbol{k}) - E_G{}^*)}. \quad (15)$$

The relation between the above equations can be expressed as

$$\begin{aligned} <A> = <E_G|A|E_G{}^*> &= \mathrm{Re}<E_G{}^*|A|E_G{}^*> \\ &- \frac{2i\Gamma^2}{\pi} \sum_{l,l'} \int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m} \left[E(\boldsymbol{k}) - E(\boldsymbol{k}')\right] \\ &\times \frac{<k', \hat{k}', l'|A|k, \hat{k}, l>}{|E(\boldsymbol{k}') - E_G|^2 |E(\boldsymbol{k}) - E_G|^2} \\ &+ \frac{4\Gamma^3}{\pi} \sum_{l,l'} \int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m} \\ &\times \frac{<k', \hat{k}', l'|A|k, \hat{k}, l>}{|E(\boldsymbol{k}') - E_G|^2 |E(\boldsymbol{k}) - E_G|^2}. \end{aligned} \quad (16)$$

This means that the result obtained in [15] is valid at leading order in Γ and that, in general, $<E_G|A|E_G{}^*> \neq \mathrm{Re}<E_G{}^*|A|E_G{}^*>$.

Hereafter we would like to discuss briefly the main differences between the present approach and the methods due to other authors, mainly [10,13,19]. The treatment of Bohm et al. [13,19] is based on the use of the Schwartz space $\mathcal{S}$ of rapidly-decreasing functions on the real axis while we have introduced the space of basic functions ξ, which are integer analytic functions. In [13] a GS is defined as a continuous linear functional of analytic functions of $\mathcal{S}$ which should vanish on a circle, in the limit of infinite radius, in the complex energy plane. This procedure requires the extension of the spectrum of H and it can yield to non-physical values. In our approach a GS is an element of ξ' and to define it we have used Dirac's formula. This definition has direct consequences upon the structure of a GS and in [17] it is shown that the integration done in [14] and the use of the support $E_0 \leq E \leq E_1$ (in the limit $E_0 \to -\infty$ and $E_1 \to +\infty$) does not yield to the same result. We think that the use of ultradistributions, i.e, the space of functions ξ', has an obvious advantage, namely: it does not require of any analytic continuation, like it is done in [10,14]. Both the energy and the normalization of a GS, as we have shown, can be obtained directly from Dirac's formula, instead.

To conclude, we have presented a mathematical representation of GS based on the RHS and calculated the expectation value of a self-adjoint operator on a resonant state. We have found that Berggren's expansion is valid at leading order in the imaginary part of the energy E_G. This result is illustrative of the adopted representation. However, a more complete comparison between the results of the present method and the one developed in [7,8], concerning the description of GS, would be useful in order to assess the advantages or the drawbacks of the use of RHS for numerical work.

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