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Systematic QRPA study of β^- and β^+/EC decay transitions to excited states of $^{110-114}$Cd and $^{114-122}$Sn ★

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Abstract

A systematic study of the β^- and β^+/EC decay transitions of odd-odd nuclei into the ground state, first excited quadrupole state (2_1^+) and the two-quadrupole-phonon triplet (0_{2ph}^+, 2_{2ph}^+ and 4_{2ph}^+) of the final even-even nuclei $^{110-114}$Cd and $^{114-122}$Sn is performed. In these cases the states of the even-even nucleus are fed both by β^- as well as β^+/EC decay transitions of the two adjacent double-odd nuclei. Such exceptional circumstances help to study the β^- and β^+/EC Gamow-Teller decay on equal footing starting from the predictions of the combined QRPA and pn-QRPA theories. In addition, the differences between the QRPA and the QTDA predictions are discussed. The present approach tests systematically the theoretical β^- and β^+/EC matrix elements and might be used for identification of states belonging to the two-quadrupole-phonon multiplet in the spectra of the cadmium and tin isotopes.

1. Introduction

The general theoretical structure of the allowed and first-forbidden beta-decay transitions is well-established [1-3]. However, the calculation of nuclear matrix elements which go into the expressions of the beta-decay half lives and log ft values is still in an evolutionary stage. The complexities in the calculation of the participant nuclear matrix elements have lead to different truncation schemes of the basic shell-model approach.

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For light nuclei the shell model is within the limits of applicability but for medium-heavy and heavy open-shell nuclei the calculational effort becomes intractable.

One convenient method for handling medium-heavy and heavy open-shell nuclei is the quasiparticle random-phase approximation (QRPA) [4]. The QRPA formalism takes into account two-quasiparticle excitations built of quasi-proton or quasi-neutron pairs. The excitation energies of an even-even nucleus are then obtained by diagonalizing the QRPA matrix [5]. The adjacent odd-odd nuclei can be treated conveniently by calculating their wave functions by the use of the charge-changing mode of the QRPA, the pn-QRPA [6,7], where the states of the odd-odd nucleus are generated by proton-neutron quasiparticle-pair excitations of the adjacent even-even ground state.

The transitions between the states of the odd-odd nucleus and the states of the adjacent even-even nucleus can be evaluated theoretically by using the double-commutator techniques introduced in Ref. [8] and assuming that both types of QRPA approaches produce the same vacuum. The double-commutator techniques can also be exploited to calculate the transition rate of the double beta decay to excited states of the daughter nucleus [9]. The theory of the allowed and first-forbidden decays, both in the β^- and β^+/EC mode, can be used to predict $\log ft$ values (and partial half lives) of the transitions to the ground state and excited states of the decay daughter.

Previous nuclear-structure studies have shown that the QRPA method is, indeed, successful in describing allowed β^+/EC transitions [10,11] as well as the quenching of the beta-decay strength in $2\nu\beta\beta$ transitions [12,13]. Subsequent studies [14-16] confirm that a rather reliable prediction of the nuclear matrix elements involved in the beta-decay and the double-beta-decay processes is possible. The studies have been performed either by using schematic nucleon-nucleon interactions [12,14,17,18] or realistic nucleon-nucleon interactions [10,11,13,15,16]. In addition, the consistency between the shell-model and the QRPA has been tested [19-22]. For completeness, also the effects of the particle-number projection in the QRPA equations has been studied [23,24].

Even after such an extensive QRPA study of the beta- and double-beta-decay properties of nuclei there remain open questions of nuclear structure concerning (i) the calculation of the $2\nu\beta\beta$ decay transitions to excited one- and two-phonon final states [9,18,25]; (ii) the testing of the predictive power of the QRPA in the case of first-forbidden decay transitions [8]; (iii) the study of single-beta-decay transitions to final states belonging to a two-quadrupole-phonon triplet [26], let alone (iv) a systematic study of these transitions both for the β^+ as well as the β^- branches of the weak decay.

In this work we try to shed some light mainly to the items (iii) and (iv) mentioned above. Our results, though, touch to some extent upon the first two items, as well. We perform a systematic study of the Gamow-Teller β^- and β^+/EC transitions to the ground state, first excited quadrupole state (2_1^+) and the two-phonon triplet ($0^+_{2\mathrm{ph}}$, $2^+_{2\mathrm{ph}}$ and $4^+_{2\mathrm{ph}}$) of the final double-even nucleus in the combined framework of the QRPA and the pn-QRPA. At the same time we use also the less sophisticated QTDA and pn-QTDA approaches to see the effect of the ground-state correlations upon the beta-decay observables. As testing region we have chosen the double-even cadmium isotopes

$^{110-114}$Cd and the double-even tin isotopes $^{114-122}$Sn. In these cases experimental data are available in combination with the possibility to follow both beta-decay branches, β^- and β^+/EC, feeding the same double-even final nucleus. In this way we try to find out which type of transitions are well described by the QRPA approach and which are the moot points of the theory. This study reflects also to the item (i) since the second virtual branch of the $2\nu\beta\beta$ transition amplitude to the excited final states [9] contains the presently studied β^- decay branch as the leading contribution.

In addition to giving information about the behaviour of the calculated nuclear matrix elements, the present study also gives a new view to the systematics of the two-quadrupole-phonon states of the Cd and Sn isotopes. In the present approach we can complement the study of electromagnetic decay properties and excitation energies of those states [27-33] with the comparison between theoretical and the experimental beta-decay $\log ft$ values.

In Section 2 the necessary theoretical framework is shortly reviewed and then applied in Section 3 where the main results are presented and analyzed. Finally, in Section 4, the conclusions are drawn.

2. Review of the formalism

To be able to describe the beta-decay transitions from the odd-odd parent ground state to the ground state and various excited states of the even-even daughter nucleus one has to generate, in a unified way, the nuclear wave functions for all the states involved in this process. This unified description of the parent and daughter states is achieved by the use of the QRPA framework in its charge-conserving formalism together with its charge-non-conserving formalism. The leading component of the correlated ground states of the QRPA and the pn-QRPA is the BCS vacuum, and in the commutator treatment (quasi-boson approximation) [8] of the β^- and the β^+/EC decay transitions one can assume the QRPA and the pn-QRPA vacua to coincide.

The ground states of the odd-odd parent nucleus are calculated by defining the pn-QRPA phonons as linear combinations of quasiproton-quasineutron excitations in the standard form [5,34]:

$$|J_i^\pi M_i\rangle = \sum_{\mathrm{pn}} \left[X_{\mathrm{pn}}(J_i^\pi,1)A^\dagger(\mathrm{pn},J_iM_i) - Y_{\mathrm{pn}}(J_i^\pi,1)\tilde{A}(\mathrm{pn},J_iM_i)\right]|\mathrm{QRPA}\rangle. \quad (1)$$

The factors $X_{\mathrm{pn}}(J_i^\pi,1)$ and $Y_{\mathrm{pn}}(J_i^\pi,1)$ are the forward- and backward-going amplitudes of the first pn-QRPA phonon of angular momentum J_i and parity π obtained by the pn-QRPA diagonalization procedure [11,12]. The operators $A^\dagger(\mathrm{pn},J_iM_i)$ $(\tilde{A}(\mathrm{pn},J_iM_i))$ create (annihilate) a pair of coupled unlike (pn) quasiparticles [8,11] obtained in the BCS calculation. In the pn-QTDA approach the backward-going amplitudes vanish and the corresponding vacuum becomes exactly the BCS vacuum, the excited states being linear combinations (with amplitudes X_{pn}) of the simple two-quasiparticle excitations.

We compute the $J_{\rm f}^{+}$ excited states of the final nucleus by using the QRPA procedure built upon pp and nn two-quasiparticle excitations [4]. The associated QRPA phonons are given in the form

$$|J_{\rm f}^{+}M_{\rm f};k\rangle=\sum_{a,a'}\left[Z_{aa'}(J_{\rm f}^{+},k)A^{\dagger}(aa',J_{\rm f}M_{\rm f})-W_{aa'}(J_{\rm f}^{+},k)\tilde{A}(aa',J_{\rm f}M_{\rm f})\right]|{\rm QRPA}\rangle, \tag{2}$$

Here k numbers the states with angular momentum $J_{\rm f}$ and positive parity. The quasiparticle pair creation and annihilation operators $A^{\dagger}$ and $\tilde{A}$ for proton–proton and neutron–neutron quasiparticle pairs are defined e.g. in Ref. [8]. Furthermore, a denotes all quantum numbers needed to specify a single-quasiparticle harmonic-oscillator state for protons ($a={\rm p}$) or neutrons ($a={\rm n}$) and the QRPA amplitudes $Z_{aa'}$ and $W_{aa'}$ are obtained by diagonalizing the RPA matrix [4]. Again, in the QTDA approach the backward-going amplitudes vanish and the vacuum becomes the BCS vacuum and the excited states are linear combinations (with amplitudes $Z_{aa'}$) of the simple two-quasiproton or two-quasineutron excitations.

Beta-decay transitions between the parent nucleus and the one-phonon and two-phonon states of the daughter nucleus are mediated by the various beta-decay operators discussed in Ref. [8]. These operators, both for the β^{-} and β^{+}/EC branches, consist of the single-particle transition matrix element and the one-body transition-density part. The one-body transition densities can be evaluated by using the quasi-particle representation of the β^{-} and β^{+} decay operators given in Refs. [9,25]. The single-particle transition densities to the one-QRPA-phonon states, along with the single-particle transition matrix elements, both for the allowed and the first-forbidden transitions, are discussed in Ref. [8].

The reduced transition probabilities to the $\sqrt{\frac{1}{2}}[2_1^{+}\otimes 2_1^{+}]_{J_{\rm f}}$ two-quadrupole-phonon states can be written in the following general form for the allowed and first-forbidden β^{-} decay transitions:

$$\begin{aligned}\beta^{-}_{J_{\rm f}LJ_{\rm i}}(u,v)\equiv(J_{\rm f}^{+}\|\hat{\beta}_L^{-}\|J_{\rm i}^{\pi})&=40\sqrt{\tfrac{1}{2}}\sqrt{(2J_{\rm f}+1)(2J_{\rm i}+1)}(-1)^{J_{\rm i}+L+1}\\&\times\sum_{\rm pnp'n'}({\rm p}\|\hat{\beta}_L^{-}\|{\rm n})\Big[u_{\rm p}v_{\rm n}Z_{\rm pp'}(2^{+},1)Z_{\rm nn'}(2^{+},1)X_{\rm p'n'}(J_{\rm i}^{\pi},1)\\&+v_{\rm p}u_{\rm n}W_{\rm pp'}(2^{+},1)W_{\rm nn'}(2^{+},1)Y_{\rm p'n'}(J_{\rm i}^{\pi},1)\Big]\begin{Bmatrix} j_{\rm p} & j_{\rm p'} & 2\\ j_{\rm n} & j_{\rm n'} & 2\\ L & J_{\rm i} & J_{\rm f}\end{Bmatrix},\end{aligned} \tag{3}$$

where the factors u and v are the usual BCS occupation factors obtained from the BCS calculation of the even–even daughter nucleus. The corresponding expression for the β^{+}/EC decay reads

$$(J_{\rm f}^{+}\|\hat{\beta}_L^{+}\|J_{\rm i}^{\pi})=(-1)^{r}\beta^{-}_{J_{\rm f}LJ_{\rm i}}(v,u), \tag{4}$$

where $r=0$ for the vector operators (V) and $r=1$ for the axial-vector operators (A).

The $\log ft$ values of both the β^- and β^+/EC transitions can be obtained from the following convention [35,36]:

$$\log ft_{\pm} = \log \left[\frac{6050 \text{ s}}{S^{\pm}/S_0^{\pm}} \right] , \tag{5}$$

where the integrated shape factors $S^{\pm}$ and $S_0^{\pm}$, for the allowed and first-forbidden transitions, are given in Ref. [8].

The electric transitions of multipolarity l from the excited state $|I_k^{\pi'} M\rangle$ of the even-even nucleus, k enumerating the states of the same $I^{\pi'}$, to the QRPA ground state $|\text{QRPA}\rangle$ are treated by calculating the reduced transition probability

$$B(\text{E}l; I_k^{\pi'} \to 0^+_{\text{g.s.}}) = \hat{I}^{-2} |(0^+_{\text{g.s.}} \| T_l^{(\text{E})} \| I_k^{\pi'})|^2, \tag{6}$$

where $\hat{I} \equiv \sqrt{2I+1}$ and $T_l^{(\text{E})}$ is the electric transition operator of multipolarity l. The above reduced matrix element can be cast into the following form:

$$\begin{aligned}(0^+_{\text{g.s.}} \| T_l^{(\text{E})} \| I_k^{\pi'}) = {} & e_{\text{eff}}^{(\text{p})} \hat{I}^{-1} \sum_{\text{p}_1,\text{p}_2} (\text{p}_2 \| T_l^{(\text{E})} \| \text{p}_1)(0^+_{\text{g.s.}} \| [c^\dagger_{\text{p}_2} \tilde{c}_{\text{p}_1}]_l \| I_k^{\pi'}) \\ & + e_{\text{eff}}^{(\text{n})} \hat{I}^{-1} \sum_{\text{n}_1,\text{n}_2} (\text{n}_2 \| T_l^{(\text{E})} \| \text{n}_1)(0^+_{\text{g.s.}} \| [c^\dagger_{\text{n}_2} \tilde{c}_{\text{n}_1}]_l \| I_k^{\pi'}), \end{aligned} \tag{7}$$

where $e_{\text{eff}}^{(\text{p})}$ ($e_{\text{eff}}^{(\text{n})}$) is the effective charge (in units of the fundamental charge e) for the protons (neutrons) and $c^\dagger_{\text{p}}$ ($c^\dagger_{\text{n}}$) creates a proton (neutron) particle on the harmonic-oscillator single-particle level $|n_{\text{p}}\ l_{\text{p}}\ \frac{1}{2}\ j_{\text{p}}\rangle$ ($|n_{\text{n}}\ l_{\text{n}}\ \frac{1}{2}\ j_{\text{n}}\rangle$). The tilde operator $\tilde{c}$ is the corresponding annihilation operator, defined as $\tilde{c}_a = (-1)^{j_a+m_a} c_{-a}$ where a denotes all the harmonic-oscillator quantum numbers either for protons or neutrons. The reduced single-particle matrix element of Eq. (7) can be obtained from Ref. [37], whereas the reduced single-particle transition densities can be expressed in the QRPA approximation as

$$(0^+_{\text{g.s.}} \| [c^\dagger_a \tilde{c}_b]_l \| I_k^{\pi'}) = \hat{I}(v_a u_b + v_b u_a) l \left[Z_{ab}(I^{\pi'} k) + W_{ab}(I^{\pi'} k) \right] , \tag{8}$$

where a and b are either proton or neutron indices in the sum of Eq. (7).

3. Results and discussion

In this work we present a systematic study of the beta-decay properties of a number of cadmium and tin isotopes. The examples are chosen such that the double-even daughter nucleus is fed both by the β^- as well as the β^+/EC decay transitions to the ground state (g.s.), the first excited quadrupole state (2^+_1) and the two-quadrupole-phonon triplet (0^+_{2ph}, 2^+_{2ph} and 4^+_{2ph}). The double feeding, mentioned above, is characteristic of 8 double-even final nuclei, namely $^{110-114}$Cd and $^{114-122}$Sn, fed by allowed (Fermi and Gamow–Teller) or first-forbidden decay transitions (six different operators, see

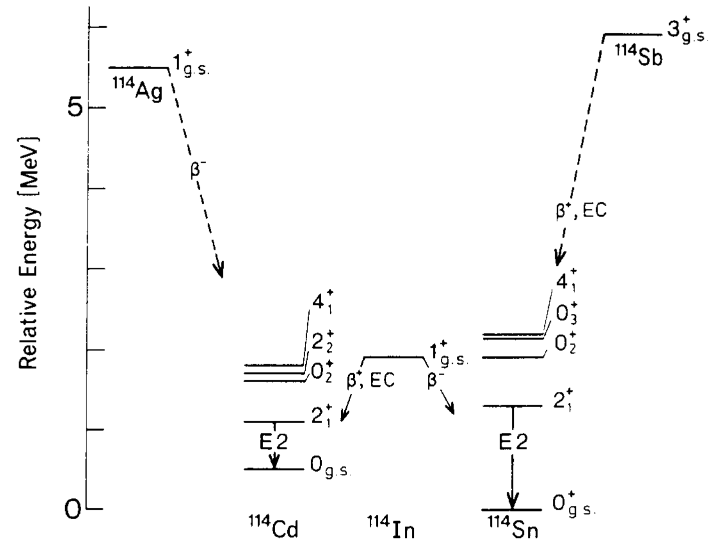

Fig. 1. A schematic drawing of the β^- and β^+/EC feeding to various levels of the ^{114}Cd and ^{114}Sn.

Ref. [8]). A representative scheme of this double feeding for both Cd and Sn is sketched in Fig. 1 in the case of mass $A = 114$.

The initial states in the double-odd systems are 1^+ states on both the β^- as well as on the β^+/EC side for the final double-even nuclei ^{114}Cd and ^{118,120}Sn. This is the optimum situation to study Gamow–Teller transitions to the final 0^+ and 2^+ states simultaneously for the β^- and β^+/EC side. In these clear-cut cases we have proceeded by adjusting the β^- transition $1^+ \rightarrow$ g.s. to data and obtaining the other β^- and β^+/EC transitions as predictions. An other class of double-even nuclei are the ones which are fed by a 1^+ state on the β^- side and a 3^+ (2^+) state on the β^+/EC side, namely ^{114}Sn and ^{116}Sn (^{110}Cd). In these cases the transitions are still of the Gamow–Teller type (or, in addition, of the Fermi type for the $2^+ \rightarrow 2^+$ transitions) but the usual considerations about the structure of the 1^+ state [11] do not necessarily apply for the description of the 2^+ or 3^+ states participant in the β^+/EC decay. The remaining two cases, namely ^{112}Cd and ^{122}Sn contain a 1^+ Gamow–Teller feeding on one side and a 2^- first-forbidden feeding on the other (in the case of 0^+ final states the transitions are unique first-forbidden transitions).

The one-phonon QRPA excitations range in character from collective phonons to pure two-quasiparticle or particle–hole (in the case of proton excitations of the tin isotopes) structures. The lowest quadrupole excitation (2^+_1) is a very collective one for the double-even nuclei considered in this article. This can be seen from the large B(E2) values of the $2^+_1 \rightarrow$ g.s. transition as discussed later. In the present QRPA order of approximation the two-phonon states are degenerate at twice the energy of the one-quadrupole-phonon state. Thus the quoted theoretical results for the beta transition strengths contain a simplifying assumption which could, in principle, be improved by taking into account

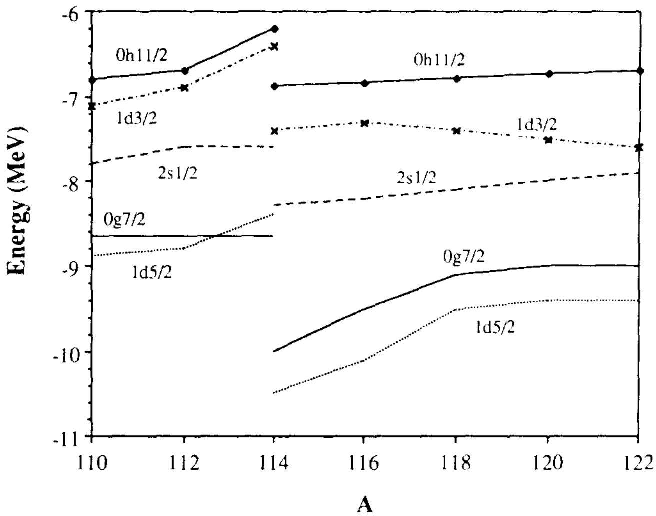

Fig. 2. Adopted neutron single-particle energies for $^{110-114}$Cd (l.h.s.) and for $^{114-122}$Sn (r.h.s).

the phonon-phonon coupling [38].

The calculations are performed combining the following four steps:

(i) The double-even ground state and the excitation spectrum of the final states are calculated using the QRPA (and the QTDA) method.

(ii) The initial states of the double-odd nuclei are generated in the pn-QRPA (and the pn-QTDA).

(iii) Various experimental data on energy observables are used to adjust the single-particle energies and the nucleon-nucleon interaction in each case separately.

(iv) The reduced matrix elements of the beta-decay transitions are treated using the double-commutator techniques of Ref. [8].

The active single-particle basis was chosen, both for protons and neutrons, to consist of oscillator major shells $3\hbar\omega$ and $4\hbar\omega$, supplemented by the $h_{11/2}$ intruder orbital from the next higher major shell, leaving the nucleus $^{40}_{20}Ca_{20}$ as the core. To first approximation, the single-particle energies were obtained from a Coulomb-corrected Woods-Saxon potential. However, the neutron single-particle energies were adjusted in the vicinity of the Fermi surface for all the even cadmium and tin isotopes to yield in a BCS calculation the observed low-energy spectrum of the adjacent odd-mass isotopes, as explained below. The resulting neutron and proton single-particle energies near the Fermi energy are depicted in Figs. 2 and 3, respectively. In particular, the proton orbitals $1d_{5/2}$ and $0g_{7/2}$ have been exchanged relative to their Woods-Saxon ordering for the Sn isotopes (see Fig. 3) to improve the description of the β^+/EC Gamow-Teller transition to the final ground state. The reference single-quasineutron data for the odd-mass cadmium isotopes ($A \leqslant 117$) are depicted in Fig. 4, where the quasiparticle energies are measured relative to the $s_{1/2}$ level. The corresponding tin data can be taken from Refs. [32,39]. For the cadmium isotopes also a single-quasiproton scheme,

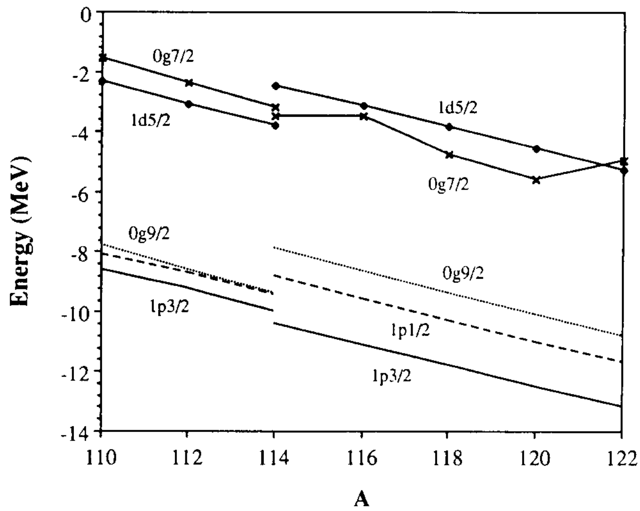

Fig. 3. Adopted proton single-particle energies for $^{110-114}$Cd (l.h.s.) and for $^{114-122}$Sn (r.h.s.).

depending only weakly on the neutron number, can be constructed. This is shown in Fig. 5, where the quasiproton energies are measured relative to the $g_{9/2}$ level.

The nucleon–nucleon interaction, used to calculate nuclear wave functions, is based on the G-matrix derived from the Bonn one-boson-exchange potential [40]. The bare G-matrix is renormalized to correspond to finite size of the nucleus [13] and the limited model space used in the calculation. The short-range nuclear correlations are renormalized by defining an overall scale of the monopole part of the interaction separately for

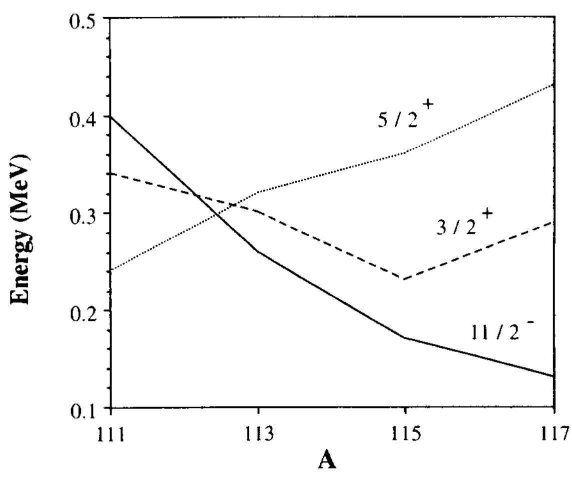

Fig. 4. Experimental neutron-quasiparticle energies relative to the $s_{1/2}$ level for $^{111-117}$Cd as taken from Ref. [45].

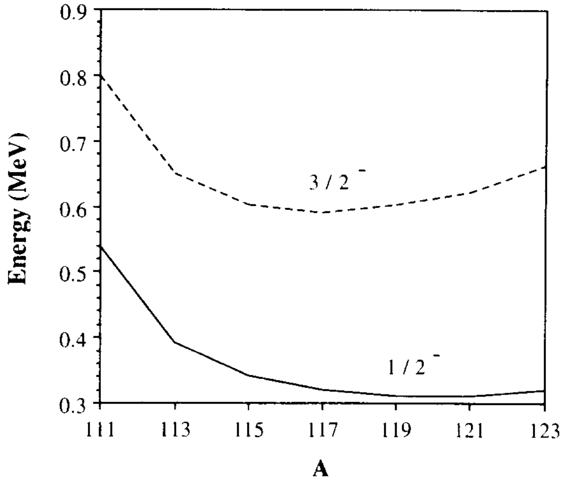

Fig. 5. Experimental proton-quasiparticle energies relative to the $g_{9/2}$ level for $^{111-123}$In as taken from Ref. [45].

protons and neutrons. Technically this is achieved by performing a BCS calculation and comparing the thus-obtained pairing gap with the one extracted from the empirical separation energies [41] in a manner described in Ref. [11].

For simplicity, the G-matrix was calculated for A = 110 and then scaled using the different interaction parameters to obtain the two-body interaction for each particular nucleus involved in the calculation. The resulting values of the scaling parameters are

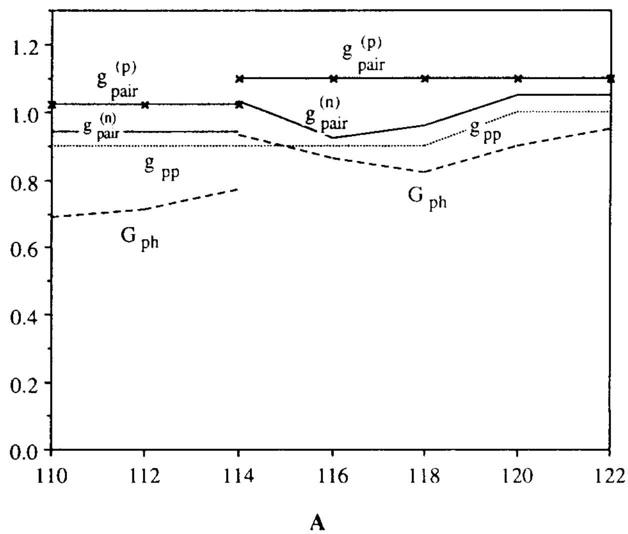

Fig. 6. Adopted values for the renormalization parameters of the G-matrix interaction for $^{110-114}$Cd (l.h.s.) and for $^{114-122}$Sn (r.h.s.).

depicted in Fig. 6, where the parameters $g^{(n)}_{\rm pair}$ and $g^{(p)}_{\rm pair}$ are the above-mentioned scaling parameters for the neutron–neutron and proton–proton pairing, respectively. The meaning and determination of the other scaling parameters shown in the figure is discussed below. Here the scaling-parameter value 1.0 corresponds to pure G-matrix interaction, and as can be seen, the deviations from this value are at most 30 per cent. This deviation could even be reduced by calculating the G-matrix separately for each different mass number A. However, the simplicity of the present systematics would then be lost without gaining any qualitative improvement of the final results.

After settling the values of the pairing parameters, two parameters remain to fix the overall scale of the proton–neutron interaction used to generate the states J^{π} of the odd–odd nucleus in the pn-QRPA calculation: the particle–hole ($g_{\rm ph}$) and the particle–particle ($g_{\rm pp}$) strength, separately for each multipolarity J^{π} [11]. In our applications we need only the lowest state of each multipolarity since the beta-decaying state is always the ground state of the initial odd–odd nucleus. For all the cases the particle–hole strength has only a small effect upon the beta-decay properties of the odd–odd nucleus at low energies. It, however, affects the energy of the 1^+ Gamow–Teller giant resonance (GTGR) in the odd–odd nucleus. The common value of $g_{\rm ph} = 1.4$ was chosen to reproduce the semiempirical location [42] of the GTGR in the cadmium and tin regions using the method of Ref. [11].

The magnitude of the particle–particle strength, $g_{\rm pp}$, in the pn channel strongly affects the theoretical $\log ft$ value of the β^- Gamow–Teller transition $1^+ \rightarrow$ g.s. in the QRPA order of approximation. Actually it is possible to fit the corresponding experimental $\log ft$ value by adjusting $g_{\rm pp}$. This would yield some 20 per cent variation around the value $g_{\rm pp} = 1.0$ corresponding to pure G-matrix interaction. Instead of adjusting $g_{\rm pp}$ we have adopted, for simplicity, a reasonable "average" value of 0.9 ($110 \leqslant A \leqslant 118$) and 1.0 ($120 \leqslant A \leqslant 122$) for this parameter as depicted in Fig. 6. Contrary to the pn-QRPA results, the results of the pn-QTDA are practically independent of $g_{\rm pp}$ and strongly overestimate the $1^+ \rightarrow$ g.s. β^- strength as discussed later in this section.

In the case of the even–even nuclei the excitation spectrum was calculated using the QRPA (and the QTDA) approach including two-quasiproton and two-quasineutron excitations. The lowest calculated 2^+ state for the Cd and Sn isotopes is quite collective, i.e. contains a large number of two-quasiproton or two-quasineutron amplitudes for the Cd isotopes and several large two-quasineutron amplitudes, but only a few non-negligible two-quasiproton amplitudes, for the Sn isotopes. The proton–proton and neutron–neutron interaction was scaled using two scaling factors, namely one ($G_{\rm ph}$) for the particle–hole interaction channel and the other ($G_{\rm pp}$) for the particle–particle interaction channel. The value of the parameter $G_{\rm ph}$ was fixed by requiring that the energy of lowest calculated 2^+ state coincides with the corresponding experimental one. As can be seen from Fig. 6 the value (0.7–0.9) of this parameter is systematically less than the bare G-matrix value $G_{\rm ph} = 1.0$. The parameter $G_{\rm pp}$ does not affect the properties of the lowest 2^+ state and thus the bare G-matrix value, $G_{\rm pp} = 1.0$, was adopted for all the calculated double-even nuclei.

After fixing the nucleon–nucleon interaction with the available spectroscopic data the

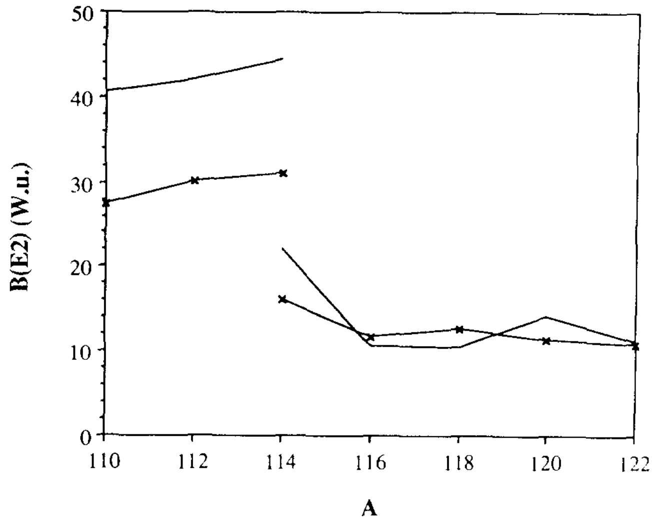

Fig. 7. Comparison of the calculated (solid line) and experimental (solid line with crosses) $B(\mathrm{E2}; 2_1^+ \to \mathrm{g.s.})$ values for $^{110-114}$Cd (l.h.s.) and for $^{114-122}$Sn (r.h.s.).

beta-decay observables are left as parameter-free predictions. In addition, the calculated reduced quadrupole decay probability $B(\mathrm{E2}; 2_1^+ \to \mathrm{g.s.})$ can be used to test the correctness of the wave function of the 2_1^+ state. To enable a comparison between the calculated $B(\mathrm{E2})$ value and the experimental one we have used the Bohr and Mottelson effective charges [37] for the theoretical proton and neutron contributions. The resulting theoretical $B(\mathrm{E2})$ values along with the experimental ones are shown in Fig. 7. One immediately notices that the experimental $B(\mathrm{E2})$ values of the Sn isotopes are nicely reproduced by the theory, whereas for the Cd isotopes the adopted effective charges overestimate the core-polarization effects.

The QTDA results for the $B(\mathrm{E2})$ values are considerably worse than the corresponding QRPA results. They underestimate the $B(\mathrm{E2})$ values yielding $B(\mathrm{E2})_{\mathrm{exp.}}/B(\mathrm{E2})_{\mathrm{th.}}$ = 2.0–2.4 for the Cd isotopes and $B(\mathrm{E2})_{\mathrm{exp.}}/B(\mathrm{E2})_{\mathrm{th.}}$ = 1.8–2.5 for the Sn isotopes. This is due to the fact that in the QRPA the forward-going (Z_{ab}) and the backward-going (W_{ab}) amplitudes sum coherently in Eq. (8) whereas the W_{ab} amplitudes are missing in the QTDA. Even though the W_{ab} amplitudes are much smaller than the corresponding Z_{ab} amplitudes their contribution is non-negligible and yield a coherent enhancement of the γ-decay strength.

The beta-decay data is shown in Tables 1–4. As already said before the $1^+ \to 0^+_{\mathrm{g.s.}}$ β^- decay is very sensitive to the value of the particle–particle strength parameter g_{pp} in the pn-QRPA calculation. This is the result of increasing ground-state correlations with increasing g_{pp} leading to increase in the magnitude of the backward-going amplitudes whose contribution thus tends to cancel the contribution coming from the forward-going amplitudes (for details see Ref. [12]). In Table 1 the theoretical values for these transitions correspond to the "average" values of 0.9 ($110 \leqslant A \leqslant 118$) and 1.0

Table 1
The β^- decay transitions from the ground state of the Ag and In isotopes to the ground state (g.s.) and the first excited quadrupole state (2_1^+) of Cd and Sn, respectively. The data are taken from Refs. [44–50] for the even masses $A = 110$–122, respectively. The theoretical numbers correspond to average g_{pp} parameter as explained in the text

Initial nucleus	J_i^π	g.s. exp.	g.s. th.	2_1^+ exp.	2_1^+ th.
^{110}Ag	1^+	4.7	5.1	5.5	5.8
^{112}Ag	2^-	9.8	11.3	8.2	8.6
^{114}Ag	1^+	5.1	5.4	5.7	5.8
^{114}In	1^+	4.5	5.2	5.6	5.2
^{116}In	1^+	4.7	4.7	5.8	5.2
^{118}In	1^+	4.7	4.3	5.5	5.3
^{120}In	1^+	5.0	4.3	5.2	5.3
^{122}In	1^+	5.1	4.3	5.4	5.4

($120 \leqslant A \leqslant 122$) for this parameter. These "global" values reproduce the experimental $\log ft$ values of the $1^+ \to 0^+_{\text{g.s.}}$ β^- transitions reasonably well and serve as an indicator of the rough magnitude of this parameter.

The rest of the transitions (to 2_1^+ and the two-phonon states) are, in general, not sensitive to the variation in the value of g_{pp}. This is due to the special structure of the one-phonon [8] and two-phonon (Eq. (3)) transition amplitudes where there are two or three small backward-going amplitudes (Y_{pn} and $W_{aa'}$) multiplied by each other thus yielding small numbers in comparison with a similar contribution coming from the forward-going amplitudes (X_{pn} and $Z_{aa'}$) which, in general, are much bigger than the corresponding backward-going amplitudes. Only in cases where the forward-going amplitudes are small and/or the BCS occupation factors u and v are extremely favourable (e.g. $u_p v_n$ is much smaller than $v_p u_n$ in Eq. (3)) the backward contribution can become important. For the nuclei under discussion the $\log ft$ values of the allowed transitions change by less than 0.6 units for the 2_1^+ decay and less than 0.5 units for the two-

Table 2
As Table 1 for the β^+/EC decay transitions In→Cd and Sb→Sn

Initial nucleus	J_i^π	g.s. exp.	g.s. th.	2_1^+ exp.	2_1^+ th.
^{110}In	2^+	–	–	6.1 [a]	5.9
^{112}In	1^+	4.7	3.6	5.3	4.9
^{114}In	1^+	4.9	3.6	> 5.2	4.7
^{114}Sb	3^+	–	–	5.2	5.8
^{116}Sb	3^+	–	–	5.2	4.9
^{118}Sb	1^+	4.5	3.6	5.8	5.7
^{120}Sb	1^+	4.5	3.5	5.6	6.0
^{122}Sb	2^-	10.6	9.5	8.9	9.4

[a] Taken from Ref. [33].

Table 3
The β^- decay transitions Ag→Cd and In→Sn: transitions to the two-quadrupole-phonon triplet 0^+_{2ph}, 2^+_{2ph} and 4^+_{2ph}. The data are taken from Refs. [44-50] for the even masses A = 110-122, respectively. The theoretical numbers correspond to average g_{pp} parameter as explained in the text

Initial		0^+_{2ph}		2^+_{2ph}		4^+_{2ph}	
nucleus	J_i^π	exp.	th.	exp.	th.	exp.	th.
^{110}Ag	1^+	$6.8(0_2^+)$, $8.1(0_3^+)$	5.3	6.9–7.4	9.4	–	–
^{112}Ag	2^-	$10.1(0_2^+)$, $11.0(0_3^+)$	12.5	8.9	8.5	9.8	9.7
^{114}Ag	1^+	$6.3(0_2^+)$, $8.1(0_3^+)$	5.5	6.5–6.9	9.0	–	–
^{114}In	$1^{+\,a}$	–	–	–	–	–	–
^{116}In	1^+	$5.9(0_2^+)$	6.2	6.3–6.4	6.8	–	–
^{118}In	1^+	$5.8(0_2^+)$, $6.9(0_3^+)$	6.1	5.5–6.0	6.8	–	–
^{120}In	1^+	$5.9(0_2^+)$, $6.8(0_3^+)$	6.0	5.8–6.3	6.6	–	–
^{122}In	1^+	$5.3(0_3^+)$, $6.5(0_2^+)$	6.2	5.7–6.1	6.8	–	–

[a] Decay from this state forbidden by the Q-value.

phonon decay due to a change of the order of 0.1 in the value of g_{pp}. For the forbidden transitions the change can be more drastic, even exceeding one unit.

The quality of the QTDA calculation for the beta-decay observables is comparable to the quality of the QRPA calculation, except in the case of the $1^+ \rightarrow 0^+_{g.s.}$ β^- transition which is predicted far too strong in the QTDA calculations, namely the $\log ft$ value for this transition ranges from 3.0 to 3.5 in the QTDA approach whereas the experimental values range from 4.5 to 5.1. Otherwise, the QTDA and the QRPA results are practically the same for the tin isotopes where essentially only the neutron quasiparticle excitations are active. For the cadmium isotopes also the proton quasiparticle excitations contribute actively and the differences between the QRPA and the QTDA results become somewhat larger but not yet significant. The same pattern is present in the excitation energies of the 2^+ states: the excitation energies are practically the same for the tins whereas differences

Table 4
As Table 3 for the β^+/EC decay transitions In→Cd and Sb→Sn

Initial		0^+_{2ph}		2^+_{2ph}		4^+_{2ph}	
nucleus	J_i^π	exp.	th.	exp.	th.	exp.	th.
^{110}In	2^+	–	–	7.0–7.5 [a]	6.3	–	–
^{112}In	1^+	$5.4(0_2^+)$	5.3	5.9	6.2	–	–
^{114}In	1^+	$5.5(0_2^+)$, $> 6.0(0_3^+)$	5.9	–	6.8	–	–
^{114}Sb	3^+	–	–	6.2–6.3	8.0	6.0–6.9	8.5
^{116}Sb	3^+	–	–	4.7 (2_3^+)	8.1	–	9.4
^{118}Sb	1^+	$5.6(0_3^+)$, $5.7(0_2^+)$	6.7	6.3–6.9	7.3	–	–
^{120}Sb	1^+	$6.0(0_3^+)$, $6.1(0_2^+)$	6.6	–	7.2	–	–
^{122}Sb	$2^{-\,b}$	–	–	–	–	–	–

[a] Taken from Ref. [33].
[b] Decay from this state forbidden by the Q-value.

exist for the cadmiums, especially in the low-energy part of the excitation spectrum.

As seen from Table 1 the β^- strength to the one-phonon states is predicted rather well by the calculation. The experimental numbers of the ground-state transitions can be reproduced exactly within the QRPA scheme but in the QTDA this not possible. However, for simplicity, the theoretical numbers for the ground-state transitions (column 4 of Table 1) correspond to a reasonable average value of the $g_{\rm pp}$ parameter ($g_{\rm pp} = 0.9$ for $110 \leqslant A \leqslant 118$, $g_{\rm pp} = 1.0$ for $120 \leqslant A \leqslant 122$), as mentioned before.

As seen in Table 2 the β^+/EC decay to the double-even ground state is predicted systematically too strong due to one or a few strong transitions between single-particle levels in the vicinity of the neutron and proton Fermi surfaces. For the β^+/EC decay transitions to the 2_1^+ state the theoretical predictions are much better. This can happen since the 1^+ ground state of odd–odd mother nucleus and the excited states of the even–even daughter are mainly connected by single-particle transitions different from the ones connecting strongly the even–even ground state with the odd–odd nucleus. In other words, the strong single-particle transitions of the ground-state decay are strongly suppressed in the decay to excited states. To improve the description of the ground-state transition one should be able to influence those parts of the odd–odd wave function that connect strongly with the QRPA vacuum. We did not succeed in doing this with reasonable single-particle basis and without altering too much the other transitions.

The β-decay transitions to members of the two-phonon triplet are depicted in Tables 3 and 4. For the cases indicated with a range of experimental $\log ft$ values (transitions to the $2^+_{\rm 2ph}$ and $4^+_{\rm 2ph}$ states) we have taken into account the data for both the second and third 2^+ state or both the first and second 4^+ state. This is due to lack of unique correspondence between theory and data on the nature of these states, i.e. it is not clear which one of the experimental levels corresponds to the theoretical two-quadrupole-phonon level. In the case of β^- transitions Table 3 shows that most of the decay transitions to the two-phonon states are described reasonably well by the QRPA, larger discrepancies existing for the decay transitions to the two-phonon levels of cadmium. Furthermore, from Table 4 one observes that mostly the theory tends to underestimate the β^+/EC transition strength, especially for the transitions from the 3^+ state. The high theoretical values ($\log ft \geqslant 8$) for some allowed transitions in Tables 3 and 4 are a result of the fact that usually there are only a few single-particle transitions which significantly contribute to the two-phonon transition amplitude and thus the location of the proton and neutron Fermi surfaces may sometimes cause a significant suppression of the product of the three forward-going amplitudes in Eq. (3). This same feature is also present in the QTDA calculation.

From Tables 1–4 one can conclude in general that the combined QRPA and pn-QRPA theories predict reasonably well most of the beta-decay properties of the studied nuclei. The comparison between experiment and theory might not be straightforward for the two-phonon states since (i) the identification of the states with two-phonon character might not be unambiguous since intruder states, i.e. states of non-spherical character, are believed [27,30,31,43] to enter the low-energy region of the studied nuclei, and (ii) we are using a simple non-interacting phonon picture without taking into account the phonon-two-quasiparticle coupling which is supposed to be significant at least for the tin isotopes [28,29,32].

Previous experimental studies of the two-quadrupole-phonon triplet [27,30,31,33] have been based primarily on level energies and electromagnetic observables. In suitable conditions the comparison between theoretical and experimental beta-decay observables might discriminate between wave functions which do and which do not fit the simple non-interacting spherical phonon picture. In this method one would need (i) systematic experimental information on the β^+/EC or β^- transitions to the low-lying J^+ (J = 0,2,4) states. In addition, (ii) the beta-decay $\log ft$ values should be discriminating enough, i.e. the beta-decay feeding should differ clearly in strength for the states of interest, say, in one unit or more in the $\log ft$ value. (ii) One has to assume that the triplet of two non-interacting phonons describes reasonably well the experimental excitations J^+ of the spherical type.

Studying Table 3 one notices that in the case of the excited 0^+ states (0_2^+ and 0_3^+ in the experimental spectrum) the criterions (i) and (ii) are roughly met by the experiment. There remains only the criterion (iii) to be full filled by the theory. To study this aspect more carefully one would need to perform a perturbative phonon-two-quasiparticle coupling calculation [28,29]. Sticking, for the time being, to our simple non-interacting phonon picture would yield the 0_2^+ state as a member of the two-phonon triplet for the Cd and Sn isotopes. In ^{112}Cd the first-forbidden β^- transition would not support this observation but, on the other hand, this very retarded transition is more liable to theoretical and experimental uncertainties than the allowed transitions from the initial 1^+ state and thus can not be considered as a good probe of the wave function. For the Sn isotopes the identification of the spherical degrees of freedom might be influenced by the indication [28,29,32] of strong phonon-two-quasiparticle coupling diminishing the predictive power of the present approach.

4. Summary and conclusions

In this article we have studied β^- and β^+/EC transitions from the ground state of an odd-odd nucleus to the ground state and excited states of the adjacent even-even nucleus. The excited final states comprise of the first quadrupole excitation (2_1^+) and the $2_1^+ \otimes 2_1^+$ two-phonon triplet (0_{2ph}^+, 2_{2ph}^+ and 4_{2ph}^+) of states. The ground state and the excited states of the even-even nucleus have been generated by using the QRPA or QTDA treatment of two-quasineutron and two-quasiproton excitations. The quasineutron and quasiproton excitations were produced by a BCS calculation in a valence space consisting of the $4\hbar\omega$ and $5\hbar\omega$ oscillator major shells. In the calculations the 2_1^+ state was described as a collective QRPA phonon and the two-phonon states as a degenerate triplet built of two 2_1^+ phonons.

The beta-decaying ground state of the odd-odd parent nucleus was obtained by the use of the pn-QRPA (pn-QTDA) approach where the wave function of the odd-odd nucleus is built of quasiproton-quasineutron excitations. The various beta-decay transitions between the ground state of the odd–odd and the ground state and various excited states of the even–even nucleus were calculated in the allowed and first-forbidden approximations using the techniques of Ref. [8]. In addition, the reduced electric quadrupole probability $B(\mathrm{E2}; 2_1^+ \to \mathrm{g.s.})$ was calculated for the double-even nucleus and compared with experimental $B(\mathrm{E2})$ data to assess the validity of the calculated wave function of the first quadrupole excitation, also used to build the two-phonon excitations.

We have calculated the above-discussed double-odd-to-double-even beta-decay transitions for a number of even–even daughter nuclei in the Cd and Sn isotopic chains. The double-odd nuclei were chosen such that they are fed by β^- transitions from one side and by β^+/EC transitions from the other. In this way one can pursue systematic studies of β^- and β^+/EC nuclear matrix elements in cases where the odd–odd QRPA (QTDA) wave function for both β-branches is generated starting from the same even–even nucleus. This enables a simultaneous study of theoretical predictability of both types of beta transitions in the same nuclear environment.

The main results and observations, emerging from the present analysis, are the following:

(i) The QRPA is able to reproduce the measured strong $B(\mathrm{E2}; 2_1^+ \to 0_{\mathrm{g.s.}}^+)$ of the Cd and Sn isotopes whereas the QTDA prediction lacks severely in strength of these transitions.

(ii) In most cases, the QRPA and the QTDA theory seem to predict rather well both the β^- and β^+/EC Gamow–Teller strength in the Cd and Sn isotopes. In the case of the QRPA the experimental allowed β^- decay strength to the ground state can be reproduced by adjusting the particle–particle interaction strength (strength parameter g_{pp}) in the pn-QRPA calculation whereas in the QTDA this is not possible leading to far too strong β^- feeding of the ground state. For the other beta-decay observables the QRPA and the QTDA results are comparable.

(iii) The allowed β^+ feeding to the ground state seems to be systematically too strong for the QRPA (and the QTDA). The strong single-particle transitions, responsible for this effect, are not present, or are suppressed, in the excited-state transitions leading to $\log ft$ values comparable to experiment.

(iv) A systematic study of the allowed beta decays could be used as a tool to identify intruder states in systematics of nuclear excitations. In the present work we have attempted such an identification for the 0^+ states in the Cd and Sn isotopes $^{110-114}$Cd, $^{116-122}$Sn. The underlying assumption is that one should be able to describe part of the low-energy excitations as two non-interacting quadrupole phonons. The deviations between the experimental and theoretical β^- and β^+/EC decay rates to the excited 0^+ states could be associated with the omitted phonon-two-quasiparticle coupling which splits the two-phonon degeneracy and could render the identification very difficult. Only a proper phonon-two-quasiparticle-coupling calculation of the spherical degrees of freedom would shed light to this problem; this, however, is outside the scope of the present work.

The advantages of the present study rely upon well-tested odd–odd wave functions and on the possibility to test the even–even one-phonon wave function by the experimental

B(E2) data. In this way we can achieve a unified description of the ground states and excited states of both the even–even and the odd–odd nuclei. The shortcomings of our approach are the lack of experimental data on β^- and/or β^+/EC transitions to excited J^+ levels of interest, and the omission of anharmonicities [51] in describing the two-quadrupole-phonon spectra.

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