ELSEVIER

Nuclear Physics A 588 (1995) 847–860

NUCLEAR PHYSICS A

Study of the correspondence between dual thermal transformations and gauge fields ★

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Received 11 November 1994; revised 16 January 1995

Abstract

The correspondence between dual thermal transformations of the thermo field dynamics (TFD) and gauge fields is studied both for abelian and non-abelian theories. It is found that the action for the TFD representation of a Dirac's lagrangian remains invariant under local transformations in k-space. The conserved charge coincides with the TFD vacuum-generator G introduced by Takahashi and Umezawa. The relationship between the rules of the TFD and the principle of gauge invariance in a thermal subspace is discussed in the context of a thermal symmetry breaking.

1. Introduction

The connection between statistical mechanics and field theory at finite temperature has been studied in a series of fundamental papers by Umezawa and co-workers [1–10]. The formal aspects of the problem and the corresponding mathematical tools are the basis of the thermo field dynamics (TFD). The theory has been discussed and reviewed more recently by other authors [11,12] and compared with other methods [13–18]. It is a very useful formalism which has already been applied to describe equilibrium and non-equilibrium physical systems [19–22]. Applications of the TFD to many body theories have also been reported [23–28].

In the framework of the TFD it exists a complete correspondence between statistical averages and expectation values of field theories at finite temperature. The TFD is based on the doubling of fermion and boson fields and on the use of constraints which are imposed to separate physical and spurious degrees of freedom. This procedure is

★ This work was partially supported by CONICET, Argentina.

SSDI 0375-9474(95)00051-8

accompanied by the introduction of a thermal symmetry and by the use of a correlated thermal vacuum [29].

Recently, a formal connection between the TFD and the concept of gauge invariance has been developed [30]. In the work of Ref. [30] the degrees of freedom introduced by the thermal doubling are absorbed by the definition of a gauge field minimally coupled to physical fields. The structure of this gauge field can be obtained from the group structure of the underlying field theory [31].

The equivalence between the rules of the TFD and the principle of gauge invariance in a thermal subspace has been shown starting from the invariance of the TFD action under local thermal transformations in k-space [31].

In this paper we continue with the study of the connection between TFD and the principle of gauge invariance. In the present context the concept of gauge invariance will be always applied in relation with the explicit invariance exhibited by transformed TFD-lagrangians. These transformed lagrangians are obtained by absorbing the degrees of freedom introduced by the thermal doubling. We have extended the correspondence established in Ref. [31] for the abelian U(1) case to a non-abelian U(N) model. The structure of the solution for the TFD vacuum is discussed starting from the concept of a conserved charge, as deduced from the invariance of the TFD action under infinitesimal thermal transformations. The concept of thermal weak equalities is reviewed and the coupling between physical and gauge fields is analyzed.

Some basic concepts of the TFD are briefly presented in Section 2. The structure of the TFD in a thermal subspace is studied in Section 3. Infinitesimal thermal transformations are introduced in the same section. Applications for different group symmetries are given in Section 4. Conclusions are drawn in Section 5. Details of the formalism are given in the Appendices A and B.

2. TFD as a gauge theory

The average values of the statistical mechanics and the expectation values of a field theory can be related, within the TFD, provided the Hilbert space is enlarged to include physical and tilde (or dual) degrees of freedom. In addition, the α degree of freedom is introduced [4] and to a particular value of α ($0 \leqslant \alpha \leqslant 1$) it is assigned a given closed time-path Green function formalism [3,17]. The TFD representation is defined by a set of operators which are bilinear in physical and dual fields [29]. From the TFD tilde substitution rule the Kubo–Martin–Schwinger (KMS) condition can be derived [2,5]. The tilde substitution rule determines the structure of the thermal vacuum.

An interesting aspect of the TFD theory is its strong similarity with a gauge theory manifested in a restricted subspace H_{th} [30]. Due to it a lagrangian representing free fermions can be expressed as a gauge invariant lagrangian which includes the coupling of the fermion fields to an effective gauge field.

The starting point of the discussion presented in Ref. [30] is the fact that in the TFD the matrix elements of the lagrangian are always invariants under gauge transformations provided the space of physical states is restricted to the subspace H_{th}. This subspace is composed by the states $|\,\text{phys}\rangle$ restricted by the condition

$$\left(a^+(\boldsymbol{k},s)a(\boldsymbol{k},s) - \tilde{a}^+(\boldsymbol{k},s)\tilde{a}(\boldsymbol{k},s)\right) |\,\text{phys}\rangle = 0. \tag{1}$$

This condition can be deduced by enforcing the gauge symmetry à la Wigner–Weyl, as is shown below.

To illustrate these postulates of the TFD we can start with the definition of the equalities which are satisfied by operators and by their dual partners, namely:

$$\langle O \rangle_{\text{th}} \cong \langle \tilde{O} \rangle_{\text{th}}, \tag{2}$$

where $\langle\rangle_{\text{th}}$ denotes matrix elements in H_{th}. These conditions are known as thermal weak equalities (t.w.e.) [30].

Let us consider the case of a free Dirac lagrangian for a massive field. We have

$$L = \tfrac{1}{2} i\bar{\psi}\gamma^{\mu} \overleftrightarrow{\partial}_{\mu} \psi - m\bar{\psi}\psi. \tag{3}$$

Introducing the dual field $\tilde{\psi}$ the TFD lagrangian reads

$$\hat{L} = L - \tilde{L} = \tfrac{1}{2} i\bar{\psi}\gamma^{\mu} \overleftrightarrow{\partial}_{\mu} \psi - m\bar{\psi}\psi + \tfrac{1}{2} i\bar{\tilde{\psi}}\gamma^{\mu *} \overleftrightarrow{\partial}_{\mu} \tilde{\psi} + m\bar{\tilde{\psi}}\tilde{\psi}. \tag{4}$$

Due to t.w.e. the lagrangian $\hat{L}$ is gauge invariant in H_{th} since [30]

$$\langle \bar{\psi}\gamma^{\mu}\psi \rangle_{\text{th}} = \langle \bar{\tilde{\psi}}\gamma^{\mu *}\tilde{\psi} \rangle_{\text{th}}. \tag{5}$$

The structure of the TFD theory and the conditions imposed to construct the subspace (1) can be understood by applying standard techniques of the gauge theory, namely: by searching for the global gauge invariance of $\hat{L}$ under the transformation of the fermion fields. Let us assume that fermion fields transform like

$$\psi' = e^{i\alpha} \psi \tag{6}$$

and similarly for the field $\tilde{\psi}$.

By keeping first order terms in α the variation of the action reads

$$\delta\hat{S}(\alpha) = -\alpha \int \partial_{\mu} \left[\bar{\psi}\gamma^{\mu}\psi - \bar{\tilde{\psi}}\gamma^{\mu *}\tilde{\psi}\right] \tag{7}$$

and it obviously vanishes. Thus the current

$$\hat{J}^{\mu} = \bar{\psi}\gamma^{\mu}\psi - \bar{\tilde{\psi}}\gamma^{*\mu}\tilde{\psi} \tag{8}$$

can be defined and from it the charge

$$\hat{Q} = \int \hat{J}^{(\mu=0)} \, d^3x \tag{9}$$

can be computed. It is given by

$$\hat{Q} = \int \left[\psi^+\psi - \tilde{\psi}^+\tilde{\psi}\right] d^3x. \tag{10}$$

By construction this charge is conserved. The connection with the TFD rule (1) can be obtained if the condition

$$\hat{Q} \mid \text{phys}\rangle = 0.$$

is imposed.

To show that it is equivalent to (1) we can expand the fermion fields ψ and $\tilde{\psi}$ in terms of a complete set of particle (antiparticle) creation (annihilation) operators and their corresponding dual operators.

The final result is given by

$$\hat{Q} = Q - \tilde{Q} = \sum_{\boldsymbol{k},s} (n(\boldsymbol{k},s) - \tilde{n}(\boldsymbol{k},s)) , \tag{11}$$

where

$$n(\boldsymbol{k},s) = a^{+}(\boldsymbol{k},s)a(\boldsymbol{k},s) - b^{+}(\boldsymbol{k},s)b(\boldsymbol{k},s) \tag{12}$$

and $\tilde{n}(\boldsymbol{k},s)$ has a similar expression in terms of tilde creation and annihilation operators. Obviously the result (11) is fully consistent with condition (1) and it shows that the TFD is a realization of a global gauge invariance. Following Ref. [30] we can introduce a gauge field A_μ such that

$$\langle -gA_\mu \bar{\psi}\gamma^\mu \psi\rangle_{\text{th}} = \langle \tfrac{1}{2} i\bar{\tilde{\psi}}\gamma^{\mu *} \overleftrightarrow{\partial}_\mu \tilde{\psi}\rangle_{\text{th}} + \langle m\bar{\tilde{\psi}}\tilde{\psi}\rangle_{\text{th}}. \tag{13}$$

Therefore it is possible to write for the fermion sector of the lagrangian the expression

$$\hat{L} \cong L_g = \tfrac{1}{2} i\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - gA_\mu \bar{\psi}\gamma^\mu \psi - m\bar{\psi}\psi \tag{14}$$

which is manifestly U(1)-invariant.

This construction can be generalized for the U(N) case as follows. Starting from the TFD lagrangian

$$\hat{L} = L - \tilde{L} = \tfrac{1}{2} i\underline{\bar{\psi}}\gamma^\mu \overleftrightarrow{\partial}_\mu \underline{\psi} - m\underline{\bar{\psi}}\,\underline{\psi} + \tfrac{1}{2} i\underline{\bar{\tilde{\psi}}}\gamma^{\mu *} \overleftrightarrow{\partial}_\mu \underline{\tilde{\psi}} + m\underline{\bar{\tilde{\psi}}}\,\underline{\tilde{\psi}}, \tag{15}$$

where $\underline{\psi}$ and $\underline{\tilde{\psi}}$ denote multiplets of N Dirac spinors (hereafter denoted as Dirac's N-plets) with components $\psi^{(n)}$, with $n = 1 \to N$, and using the t.w.e.

$$\langle -gA_\mu \cdot \underline{\bar{\psi}}\gamma^\mu \boldsymbol{T}\underline{\psi}\rangle_{\text{th}} = \langle \tfrac{1}{2} i\underline{\bar{\tilde{\psi}}}\gamma^{\mu *} \overleftrightarrow{\partial}_\mu \underline{\tilde{\psi}}\rangle_{\text{th}} + \langle m\underline{\bar{\tilde{\psi}}}\,\underline{\tilde{\psi}}\rangle_{\text{th}}, \tag{16}$$

where $\boldsymbol{T}$ are the generators of U(N), thus

$$\hat{L} \cong L_g = \tfrac{1}{2} i\underline{\bar{\psi}}\gamma^\mu \overleftrightarrow{\partial}_\mu \underline{\psi} - g\boldsymbol{A}_\mu \underline{\bar{\psi}}\gamma^\mu \boldsymbol{T}\underline{\psi} - m\underline{\bar{\psi}}\,\underline{\psi}. \tag{17}$$

Therefore, the TFD tilde-degrees of freedom have been replaced by the coupling of the original fermionic current with a vector potential. In the present work we are focussing on the converse, namely: on the conditions which should be imposed on a gauge theory in order to describe thermal effects in some subspace, as we shall discuss in the next section.

3. Infinitesimal thermal transformations

In the following we shall show how to construct the TFD vacuum starting from
i) the definition of infinitesimal thermal Bogoliubov transformations,
ii) the definition of a TFD conserved charge from the introduction of the dual thermal action and by using Noether's theorem and
iii) the explicit construction of H_{th}.

Let us start with an abelian theory. The Dirac lagrangian for a free massive fermion field, L, and its TFD representation, $\hat{L}$, are given by Eqs. (3) and (4), respectively.

The field ψ and $\tilde{\psi}$ can be expanded as

$$\psi(x) = \psi_a(x) + \psi_b^{+\mathrm{t}}(x), \tag{18}$$

where we have used a compact notation for the particle sector of the field

$$\psi_a(x) = \frac{1}{\sqrt{V}} \sum_k \left(\frac{m}{\omega}\right)^{1/2} \sum_{s=\pm} \left[a(k,s) U(k,s)\, \mathrm{e}^{-ik\cdot x}\right],$$

as well as for the anti-particle sector

$$\psi_b^{+\mathrm{t}}(x) = \frac{1}{\sqrt{V}} \sum_k \left(\frac{m}{\omega}\right)^{1/2} \sum_{s=\pm} \left[b^+(k,s) V(k,s)\, \mathrm{e}^{ik\cdot x}\right],$$

respectively.

In the above expressions the operators $a(k,s)$ ($b(k,s)$) are particle (antiparticle) operators, $U(k,s)$ and $V(k,s)$ are spinors of four components and the superscript t denotes transposition.

Similarly, the dual field is written

$$\tilde{\psi}(x) = \tilde{\psi}_a(x) + \tilde{\psi}_b^{+\mathrm{t}}(x). \tag{19}$$

By Fourier-transforming these components one can define the fermion fields in momentum space. They will be denoted by $\psi_a(k)$, $\psi_b^{+\mathrm{t}}(k)$, $\tilde{\psi}_a(k)$ and $\tilde{\psi}_b^{+\mathrm{t}}(k)$, respectively.

The inverse temperature β can be introduced by transforming the fields $\psi_a(k)$, $\psi_b^{+\mathrm{t}}(k)$ and the dual (tilde) fields. For the infinitesimal thermal transformation we have chosen a local-abelian form in (k,β)-space

$$U(\Theta) = \mathrm{e}^{-i\alpha\sigma^{(2)}\Theta_{\pm}(k,\beta)}, \tag{20}$$

where α is an infinitesimal real number, $\Theta_{\pm}(k,\beta)$ is a scalar function and $\sigma^{(2)}$ is a Pauli matrix

$$\sigma^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(the notation of Bjorken and Drell [32] for Lorentz and spin-indices and the convention for the metric tensor are adopted).

The index $\pm$ indicates that the function Θ has different values for particle $(+)$ and antiparticle $(-)$ states. The transformed fields are given by

$$\begin{pmatrix} \psi_a(k,\beta) \\ \tilde{\psi}_a^{+t}(k,\beta) \end{pmatrix} = e^{-i\alpha\sigma^{(2)}\Theta_+(k,\beta)} \begin{pmatrix} \psi_a(k) \\ \tilde{\psi}_a^{+t}(k) \end{pmatrix},$$

$$\begin{pmatrix} \psi_b(k,\beta) \\ \tilde{\psi}_b^{+t}(k,\beta) \end{pmatrix} = e^{-i\alpha\sigma^{(2)}\Theta_-(k,\beta)} \begin{pmatrix} \psi_b(k) \\ \tilde{\psi}_b^{+t}(k) \end{pmatrix}. \tag{21}$$

The TFD action $\hat{S} = \int \hat{L}\, d^4x$ is invariant under this transformation. To show it the fields (21) are expanded at leading order in the parameter α and transformed back to coordinates. The resulting components are

$$\begin{aligned}
\psi_a(x,\beta) &= \psi_a(x) - \alpha F^{-1}\{\Theta_+(k,\beta)\} \star \tilde{\psi}_a^{+t}(x), \\
\tilde{\psi}_a^{+t}(x,\beta) &= \tilde{\psi}_a^{+t}(x) + \alpha F^{-1}\{\Theta_+(k,\beta)\} \star \psi_a(x), \\
\psi_b(x,\beta) &= \psi_b(x) - \alpha F^{-1}\{\Theta_-(k,\beta)\} \star \tilde{\psi}_b^{+t}(x), \\
\tilde{\psi}_b^{+t}(x,\beta) &= \tilde{\psi}_b^{+t}(x) + \alpha F^{-1}\{\Theta_-(k,\beta)\} \star \psi_b(x),
\end{aligned} \tag{22}$$

where $\star$ indicates the convolution operation.

The functional form of $\Theta_\pm(k,\beta)$ is given by

$$\sin^2(\Theta_\pm) = \frac{1}{1 + e^{\beta(|k^0| \mp \mu)}} \tag{23}$$

and it guarantees that the product of infinitesimal thermal transformations is a thermal Bogoliubov transformation. The transformed lagrangian $\hat{L}$ is a function of the form

$$\hat{L} \to \hat{L} + \alpha\hat{l}(\beta) + O(\alpha^2). \tag{24}$$

Neglecting terms of order $O(\alpha^2)$ the action is written as

$$\hat{S} \to \hat{S} + \alpha \int \hat{l}(\beta)\, d^4x. \tag{25}$$

The term $\alpha \int \hat{l}(\beta)\, d^4x$ has a rather involved structure. To write it down in a compact form we have defined the auxiliary fields

$$\begin{aligned}
\tilde{\chi}(x,\beta) &= \Theta_+(x,\beta) \star \tilde{\psi}_a^{+t}(x) + \Theta_-(x,\beta) \star \tilde{\psi}_b(x), \\
\chi(x,\beta) &= \Theta_+(x,\beta) \star \psi_a^{+t}(x) + \Theta_-(x,\beta) \star \psi_b(x),
\end{aligned} \tag{26}$$

where $\Theta_\pm(x,\beta)$ is a short-hand notation for the inverse Fourier transform of $\Theta_\pm(k,\beta)$. The expression for $\int \hat{l}(\beta)\, d^4x$ becomes:

$$\begin{aligned}
\int \hat{l}(\beta)\, d^4x = \int & i\partial_\mu\bar{\psi}(x)\gamma^\mu\tilde{\chi}(x,\beta) - i\bar{\tilde{\chi}}(x,\beta)\gamma^\mu\partial_\mu\psi(x) \\
& + m\bar{\psi}(x)\tilde{\chi}(x,\beta) + m\bar{\tilde{\chi}}(x,\beta)\psi(x) - i\partial_\mu\bar{\tilde{\psi}}(x)\gamma^{\mu *}\chi(x,\beta) \\
& + i\bar{\chi}(x,\beta)\gamma^{\mu *}\partial_\mu\tilde{\psi}(x) + m\bar{\tilde{\psi}}(x)\chi(x,\beta) + m\bar{\chi}(x,\beta)\tilde{\psi}(x).
\end{aligned} \tag{27}$$

Due to the equation of motion for ψ and $\tilde{\psi}$

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = \left(i\gamma^{\mu *}\partial_{\mu} + m\right)\tilde{\psi}(x) = 0, \tag{28}$$

the integral (27) vanishes independently of β and the action $\hat{S}$ remains invariant. The use of Noether's theorem yields a conserved charge Q_{TFD} and its structure is determined by performing the variation

$$\delta\hat{S} = \int \partial_{\mu}\left(\frac{\partial\hat{L}}{\partial(\partial_{\mu}\phi_c)}\delta\phi_c\right), \tag{29}$$

where ϕ_c represents each one of the fields ψ, $\tilde{\psi}$, $\bar{\psi}$, $\bar{\tilde{\psi}}$.

This variation can be written in terms of the auxiliary fields χ and $\tilde{\chi}$ as

$$\delta\hat{S} = \alpha\int \partial_{\mu}\Big[-\tfrac{1}{2}i\bar{\psi}(x)\gamma^{\mu}\tilde{\chi}(x,\beta) + \tfrac{1}{2}i\bar{\tilde{\chi}}(x,\beta)\gamma^{\mu}\psi(x)$$
$$+\tfrac{1}{2}i\bar{\tilde{\psi}}(x)\gamma^{\mu *}\chi(x,\beta) - \tfrac{1}{2}i\bar{\chi}(x,\beta)\gamma^{\mu *}\tilde{\psi}(x)\Big]\,\mathrm{d}^4x. \tag{30}$$

The quantity in the square bracket is the current J_{μ}; hence the corresponding conserved charge can be defined by

$$Q_{\mathrm{TFD}} = \int J^0(x^0,x)\,\mathrm{d}^3x. \tag{31}$$

To compute Eq. (31) we have replaced the fields entering in Eq. (30) by the expansion (18). The result is

$$Q_{\mathrm{TFD}} = i\sum_{\boldsymbol{k}}\sum_{s=\pm}\Big[\Theta_{+}(\omega)\left(a^{+}(\boldsymbol{k},s)\tilde{a}^{+}(\boldsymbol{k},s) - \tilde{a}(\boldsymbol{k},s)a(\boldsymbol{k},s)\right)$$
$$+\Theta_{-}(\omega)\left(b^{+}(\boldsymbol{k},s)\tilde{b}^{+}(\boldsymbol{k},s) - \tilde{b}(\boldsymbol{k},s)b(\boldsymbol{k},s)\right)\Big], \tag{32}$$

where $\Theta_{\pm}(\omega)$ is defined as $\Theta_{\pm}(\mid k^0 \mid = \omega, \beta)$.

The formal relation between the structure of this conserved charge and the generator of the TFD vacuum has been already shown in Ref. [31]. An interesting question is the meaning of the spontaneous symmetry breaking of the thermal symmetry. To illustrate this point more clearly we have computed the fermionic current $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ by acting upon the fields with thermal Bogoliubov transformations [11,23–25] and by normal-ordering the operator products with respect to the thermal vacuum. The result is given by

$$\begin{aligned}\langle j^{\mu}\rangle &= 0 && \text{for } \mu = 1,2,3,\\ &= \frac{1}{V}\sum_{\boldsymbol{k},s}\left(\sin^2\Theta_{+}(\omega) - \sin^2\Theta_{-}(\omega)\right) && \text{for } \mu = 0.\end{aligned} \tag{33}$$

This regularized value for the current is the source of the equation which defines the average value of the gauge field A^{μ}

$$\Box\langle A^{\mu}\rangle = g\langle j^{\mu}\rangle. \tag{34}$$

The standard field theory leads to a vanishing expectation value of the fermionic current. In the present context, that is to say by enforcing the t.w.e., this result can be interpreted as a consequence of a spontaneous symmetry breaking. Thus, within the scope of the t.w.e., the gauge field can be interpreted as a massive one. The equation which determines the gauge field in presence of the spontaneous symmetry breaking is given by

$$\left(\Box + M^2\right)\langle A_0\rangle = g\langle j_0{}^{\mathrm{eff}}\rangle, \qquad \langle A_i\rangle = 0, \qquad i = 1,2,3. \tag{35}$$

The current j_μ^{eff} includes the screening due to the spontaneous symmetry breaking. The expression for M is obtained from the system of equations

$$\left(i\gamma^\mu\partial_\mu - g\gamma^\mu A_\mu - m\right)\psi = 0, \qquad \partial^\nu F_{\nu\mu} - g\bar{\psi}\gamma_\mu\psi = 0. \tag{36}$$

The fermionic current can be written

$$\bar{\psi}\gamma_\mu\psi = \frac{i}{2m}\left(\bar{\psi}\gamma_\mu\gamma^\nu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\nu\gamma_\mu\psi\right) - \frac{g}{m}\bar{\psi}\psi A_\mu \tag{37}$$

and together with Eq. (34) it leads to

$$\Box A_\mu + \frac{g^2}{m}\bar{\psi}\psi A_\mu = \frac{ig}{2m}\left(\bar{\psi}\gamma_\mu\gamma^\nu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\nu\gamma_\mu\psi\right), \tag{38}$$

where the Lorentz gauge has been used. Noticing that

$$\langle\bar{\psi}\gamma_\mu\psi\rangle_{\mathrm{th}} = \left\langle \frac{i}{2m}\left(\bar{\psi}\gamma_\mu\gamma^\nu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\nu\gamma_\mu\psi\right)\right\rangle_{\mathrm{th}}, \tag{39}$$

it follows that

$$\langle\bar{\psi}\psi A_\mu\rangle_{\mathrm{th}} = 0. \tag{40}$$

This last equality can be worked out by using a theorem [33] on chronological products of the first type

$$\langle\bar{\psi}\psi A_\mu\rangle_{\mathrm{th}} = \langle\bar{\psi}\psi\rangle_{\mathrm{th}}\langle A_\mu\rangle_{\mathrm{th}} + \langle B_\mu\rangle_{\mathrm{th}} = 0, \tag{41}$$

where B_μ symbolize all the remaining terms appearing in the complete expression. We are now in position to evaluate the equation of motion for the mean value of A_μ. Using Eqs. (37)–(41) we have obtained

$$\Box\langle A_0\rangle_{\mathrm{th}} + \frac{g^2}{m}\langle\bar{\psi}\psi\rangle_{\mathrm{th}}\langle A_0\rangle_{\mathrm{th}} = g\langle j_0^{\mathrm{eff}}\rangle_{\mathrm{th}}. \tag{42}$$

From Eq. (35) one gets

$$M^2 = \frac{g^2}{m}\langle\bar{\psi}\psi\rangle_{\mathrm{th}} = \frac{2g^2}{V}\sum_k \frac{1}{\omega}\left(\sin^2\Theta_+(\omega) + \sin^2\Theta_-(\omega)\right). \tag{43}$$

The structure of M resembles the result corresponding to the screening of the electromagnetic field in a superconductor [34,35].

4. Extensions to the U(N) and scalar cases

For the case of a lagrangian with a $U(N)$ symmetry we have adopted the form (15) and the corresponding infinitesimal thermal transformation is given by

$$U(\Theta) = \mathrm{e}^{-i\alpha\sigma^{(2)}\otimes[\Theta_\pm]}, \tag{44}$$

where the symbol $\otimes$ denotes the tensor product, the exponent $[\Theta_\pm]$ is a diagonal N-dimensional matrix and α is a real infinitesimal. The elements $\Theta_{\pm(n,n)}$ are scalar functions of $(|k^0|, \beta)$. The particle and antiparticle components of the N-plets are denoted by $(\underline{\psi_a}(x))$ and $(\underline{\psi_b}^{+\mathrm{t}}(x))$, respectively. The same notation is adopted for the dual fields $\underline{\tilde{\psi}}$,

$$\begin{pmatrix} \underline{\psi_a}(k,\beta) \\ \underline{\tilde{\psi}_a}^{+\mathrm{t}}(k,\beta) \end{pmatrix} = \mathrm{e}^{-i\alpha\sigma^{(2)}\otimes[\Theta_+(k,\beta)]} \begin{pmatrix} \underline{\psi_a}(k) \\ \underline{\tilde{\psi}_a}^{+\mathrm{t}}(k) \end{pmatrix},$$
$$\begin{pmatrix} \underline{\psi_b}(k,\beta) \\ \underline{\tilde{\psi}_b}^{+\mathrm{t}}(k,\beta) \end{pmatrix} = \mathrm{e}^{-i\alpha\sigma^{(2)}\otimes[\Theta_-(k,\beta)]} \begin{pmatrix} \underline{\psi_b}(k) \\ \underline{\tilde{\psi}_b}^{+\mathrm{t}}(k) \end{pmatrix}. \tag{45}$$

Following the arguments introduced in the previous section the components of the transformed fields can be written in terms of auxiliary fields and expanded at leading order in the parameter α. The details are given in Appendix A. The final result for the conserved TFD charge of the $U(N)$ case is given by

$$Q_{\mathrm{TFD}} = i\sum_{n=1}^{N}\sum_{\boldsymbol{k}}\sum_{s=\pm}\left[\Theta_{+(nn)}(\omega)\left(a_n^+(\boldsymbol{k},s)\tilde{a}_n^+(\boldsymbol{k},s) - \tilde{a}_n(\boldsymbol{k},s)a_n(\boldsymbol{k},s)\right)\right.$$
$$\left. +\Theta_{-(nn)}(\omega)\left(b_n^+(\boldsymbol{k},s)\tilde{b}_n^+(\boldsymbol{k},s) - \tilde{b}_n(\boldsymbol{k},s)b_n(\boldsymbol{k},s)\right)\right]. \tag{46}$$

The current

$$j_\mu^a = \underline{\bar{\psi}}\gamma_\mu T^a \underline{\psi} \tag{47}$$

acquires a non-vanishing expectation value only for its $\mu = 0$ component, which is given by

$$\langle j_0^a \rangle = \frac{2}{V}\sum_{k,l}\left(\sin^2\Theta_{+l}(\omega) - \sin^2\Theta_{-l}(\omega)\right)T_{ll}^a. \tag{48}$$

The equation of motion for the gauge field is

$$\Box A^\mu = g j^\mu - ig\left[A_\nu, F^{\nu\mu} - \partial^\nu A^\mu\right], \tag{49}$$

where

$$A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a T_a \quad \text{and} \quad j_\mu = \sum_{a=1}^{N^2-1} j_\mu^a T_a.$$

Eq. (49) can be solved by keeping the first term of the r.h.s. as the source for the mean value of the gauge field. This procedure is allowed in presence of the spontaneous symmetry breaking of the thermal symmetry and neglecting the mean value of the commutator which appears in the r.h.s. of the above equation.

For each index a the mass M_a reads

$$M_a^2 = \frac{g^2}{m}\langle\underline{\bar{\psi}}(T^a)^2\underline{\psi}\rangle_{\mathrm{th}} = \frac{2g^2}{V}\sum_{n,k}\frac{1}{\omega}\left(\sin^2\Theta_{+(nn)}(\omega) + \sin^2\Theta_{-(nn)}(\omega)\right)(T^a)^2_{(nn)}. \tag{50}$$

Following the arguments already discussed for the abelian case the spontaneous symmetry breaking of the thermal symmetry leads to a screening mechanism of the fermionic current [34,35] and the gauge field acquires a mass; i.e., a non-vanishing mean value.

Similar conclusions can be drawn from the analysis of a scalar field. The lagrangian for this field is

$$L = \tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right). \tag{51}$$

Introducing the dual field $\tilde{\phi}$ we have for the TFD lagrangian the expression

$$\hat{L} = L - \tilde{L} = \tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \tfrac{1}{2}\left(\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - m^2\tilde{\phi}^2\right). \tag{52}$$

By following the steps outlined for the previous cases the TFD conserved charge is given by

$$Q_{\mathrm{TFD}} = \int J^0(x^0, x)\,\mathrm{d}^3x. \tag{53}$$

The intermediate steps of the derivation are given in Appendix B. The final result is given by the expression

$$Q_{\mathrm{TFD}} = i\sum_{k}\left[\Theta(\omega)\left(a^+(k)\tilde{a}^+(k) - \tilde{a}(k)a(k)\right)\right]. \tag{54}$$

Again for this case the conserved TFD charge has the structure of the TFD vacuum generator.

5. Conclusions

In this paper we have discussed the gauge structure of the TFD. The formalism is based on the substitution of a portion of the TFD lagrangian by a minimal coupling between the original fermion fields and an induced gauge field.

The connection between the gauge invariant form of the lagrangian in the thermal subspace H_{th} and the full TFD lagrangian is established by computing the screening of the induced gauge field. This has been done by introducing infinitesimal thermal transformations and the breaking of the TFD symmetry.

The structure of the transformation rules, equations of motion and the source terms of the induced gauge field, which are obtained by transforming the fields with the set of infinitesimal thermal transformations discussed in the text, suggest that the TFD vacuum can be interpreted as a condensate. The explicit structure of this condensate, in terms of original and dual fermion (or boson) fields, can be obtained from the conserved charge associated to the above mentioned thermal transformations. We have also shown that this conserved charge coincides with the vacuum generator G of the TFD.

To conclude we think that the TFD can be interpreted from a less axiomatic point of view by exploring the gauge structure of the theory. It becomes evident from the examples of thermal transformations studied in this work.

Appendix A

In this appendix we are presenting some of the intermediate steps of the treatment of the U(N) case. The fermion fields (45) are expanded, at first order in the parameter α,

$$\begin{aligned}
\underline{\psi_a}(k,\beta) &= \underline{\psi_a}(k) - \alpha([\Theta_+(k,\beta)]\underline{\tilde{\psi}_a}^{+t}(k)),\\
\underline{\tilde{\psi}_a}^{+t}(k,\beta) &= \underline{\tilde{\psi}_a}^{+t}(k) + \alpha([\Theta_+(k,\beta)]\underline{\psi_a}(k)),\\
\underline{\psi_b}(k,\beta) &= \underline{\psi_b}(k) - \alpha([\Theta_-(k,\beta)]\underline{\tilde{\psi}_b}^{+t}(k)),\\
\underline{\tilde{\psi}_b}^{+t}(k,\beta) &= \underline{\tilde{\psi}_b}^{+t}(k) + \alpha([\Theta_-(k,\beta)]\underline{\psi_b}(k)).
\end{aligned}$$

Then, these fields $\underline{\psi}(x,\beta)$ and $\underline{\tilde{\psi}}(x,\beta)$ are transformed back to coordinate space

$$\begin{aligned}
\underline{\psi_a}(x,\beta) &= \underline{\psi_a}(x) - \alpha F^{-1}\{[\Theta_+(k,\beta)]\}\star\underline{\tilde{\psi}_a}^{+t}(x),\\
\underline{\tilde{\psi}_a}^{+t}(x,\beta) &= \underline{\tilde{\psi}_a}^{+t}(x) + \alpha F^{-1}\{[\Theta_+(k,\beta)]\}\star\underline{\psi_a}(x),\\
\underline{\psi_b}(x,\beta) &= \underline{\psi_b}(x) - \alpha F^{-1}\{[\Theta_-(k,\beta)]\}\star\underline{\tilde{\psi}_b}^{+t}(x),\\
\underline{\tilde{\psi}_b}^{+t}(x,\beta) &= \underline{\tilde{\psi}_b}^{+t}(x) + \alpha F^{-1}\{[\Theta_-(k,\beta)]\}\star\underline{\psi_b}(x).
\end{aligned}$$

The operation $\star$ indicates the matrix convolution operation defined as

$$F^{-1}\{[\Theta_+(k,\beta)]\}\star\underline{\psi_a}(x) = \begin{pmatrix} F^{-1}\{\Theta_{+(11)}(k,\beta)\}\star\psi_a^{(1)}(x)\\ F^{-1}\{\Theta_{+(22)}(k,\beta)\}\star\psi_a^{(2)}(x)\\ \vdots \\ F^{-1}\{\Theta_{+(NN)}(k,\beta)\}\star\psi_a^{(N)}(x)\end{pmatrix}.$$

A similar expression holds for the transformation of the antiparticle components which are transformed with Θ_- instead of Θ_+. The value of $\Theta_{\pm(nn)}(k,\beta)$ is

$$\sin^2(\Theta_{\pm(nn)}) = \frac{1}{1+e^{\beta(|k^0|\mp\mu_n)}}.$$

Explicit expressions for the transformed lagrangian and the associated action are similar to the ones given by Eqs. (24)-(25).

The auxiliary fields, for this case, are written as

$$\underline{\tilde{\chi}}(x,\beta) = [\Theta_+(x,\beta)] \star \underline{\tilde{\psi}_a}^{+t}(x) + [\Theta_-(x,\beta)] \star \underline{\tilde{\psi}_b}(x),$$
$$\underline{\chi}(x,\beta) = [\Theta_+(x,\beta)] \star \underline{\psi_a}^{+t}(x) + [\Theta_-(x,\beta)] \star \underline{\psi_b}(x),$$

where $[\Theta_\pm(x,\beta)]$ is a short-hand notation for the inverse Fourier transform of the matrix $[\Theta_\pm(k,\beta)]$.

The expression for the term $\int l(\hat{\beta})\, \mathrm{d}^4x$ resulting from the variation of the action, cf. Eq. (25), is similar to the one given in Eq. (27).

The variation of the transformed action vanishes identically. The proof is straightforward and it is based on the use of the equation of motion of the fields. The steps leading to the expression of the conserved charge coincide with the ones given in Eqs. (30)-(32).

Appendix B

The scalar fields ϕ and $\tilde{\phi}$ of Eq. (52) are expanded in a plane wave basis

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{k} \frac{1}{\sqrt{2\omega}} \left(a(\boldsymbol{k})\,\mathrm{e}^{-ik\cdot x} + a^+(\boldsymbol{k})\,\mathrm{e}^{ik\cdot x}\right),$$
$$\tilde{\phi}(x) = \frac{1}{\sqrt{V}} \sum_{k} \frac{1}{\sqrt{2\omega}} \left(\tilde{a}(\boldsymbol{k})\,\mathrm{e}^{ik\cdot x} + \tilde{a}^+(\boldsymbol{k})\,\mathrm{e}^{-ik\cdot x}\right).$$

The corresponding local infinitesimal thermal transformations are given by

$$U(\Theta) = \mathrm{e}^{-\alpha\sigma^{(1)}\Theta(k,\beta)},$$

where α is an infinitesimal real number and

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Under $U(\Theta)$ the thermal doublets transform as

$$\begin{pmatrix} \phi(k,\beta) \\ \tilde{\phi}(k,\beta) \end{pmatrix} = \mathrm{e}^{-\alpha\sigma^{(1)}\Theta(k,\beta)} \begin{pmatrix} \phi(k) \\ \tilde{\phi}(k) \end{pmatrix},$$

where $\phi(k)$, $\tilde{\phi}(k)$, $\phi(k,\beta)$ and $\tilde{\phi}(k,\beta)$ are the Fourier transforms of $\phi(x)$, $\tilde{\phi}(x)$, $\phi(x,\beta)$ and $\tilde{\phi}(x,\beta)$, respectively.

The transformed fields read, at leading order in the parameter α,

$$\phi(k,\beta) = \phi(k) - \alpha(\Theta(k,\beta)\tilde{\phi}(k)),$$
$$\tilde{\phi}(k,\beta) = \tilde{\phi}(k) - \alpha(\Theta(k,\beta)\phi(k))$$

and the β-dependent fields are given by

$$\phi(x,\beta) = \phi(x) - \alpha F^{-1}\{\Theta(k,\beta)\} \star \tilde{\phi}(x),$$
$$\tilde{\phi}(x,\beta) = \tilde{\phi}(x) - \alpha F^{-1}\{\Theta(k,\beta)\} \star \phi(x).$$

The functional form of $\Theta(k,\beta)$ is

$$\sinh^2(\Theta) = \frac{1}{e^{\beta|k^0|} - 1}.$$

Similarly to the case of fermions this form for $\Theta(k,\beta)$ guarantees that the product of infinitesimal thermal transformations is a thermal Bogoliubov transformation.

The auxiliary fields are defined, for this case,

$$\tilde{\varphi}(x,\beta) = \Theta(x,\beta) \star \tilde{\phi}(x)$$

and

$$\varphi(x,\beta) = \Theta(x,\beta) \star \phi(x),$$

where $\Theta(x,\beta)$ is a short-hand notation for the inverse Fourier transform of $\Theta(k,\beta)$. Thus the expression for $\int \hat{l}(\beta)\, d^4x$ becomes

$$\int \hat{l}(\beta)\, d^4x = \int \Box\phi(x)\tilde{\varphi}(x,\beta) + m^2\phi(x)\varphi(x,\beta) - \Box\,\tilde{\phi}(x)\varphi(x,\beta) - m^2\tilde{\phi}(x)\varphi(x,\beta).$$

From the equation of motion for ϕ and $\tilde{\phi}$

$$\left(\Box + m^2\right)\phi(x) = 0,$$

the above defined integral vanishes independently of β. The use of Noether's theorem yields the conserved charge Q_{TFD} of Eq. (54).

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