ELSEVIER

Nuclear Physics A 589 (1995) 445–459

NUCLEAR PHYSICS A

Double-beta decay to excited states in ^{150}Nd

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Received 1 January 1995

Abstract

The pseudo SU(3) model is used to study the double-beta decay of ^{150}Nd to the ground and excited states of ^{150}Sm. Low-lying collective excitations of ^{150}Sm and its BE(2) intensities are well reproduced. Expressions for the nuclear matrix elements of the two-neutrino double-beta decay to excited states are developed and used to describe the decay of ^{150}Nd. The existence of selection rules which strongly restricts the decay is discussed.

1. Introduction

The neutrinoless double-beta decay ($\beta\beta_{0\nu}$), undetected up to now, provides the more stringent limits to the Majorana mass of the neutrino $\langle m_{\nu_c} \rangle \leqslant 1.1$ eV [1]. Its detection would imply an indisputable evidence of physics beyond the standard model and would be useful in order to select grand unification theories [2].

Theoretical nuclear matrix elements are needed to convert experimental half-life limits, which are available for many $\beta\beta$-unstable isotopes [3], into constrains for particle-physics parameters such as the effective Majorana mass of the neutrino and the contribution of right-handed currents to the weak interactions. Thus, these matrix elements are essential to understand the underlying physics.

The two-neutrino mode of the double-beta decay ($\beta\beta_{2\nu}$) is allowed as a second-order process in the standard model. It has been detected in nine nuclei [3] and has served as a test of a variety of nuclear models. The calculation of the $\beta\beta_{2\nu}$ and $\beta\beta_{0\nu}$ matrix

SSDI 0375-9474(95)00090-9

elements requires different theoretical methods. Therefore a successful prediction of the former cannot be considered a rigorous test of the latter, but gives some confidence. However, it is the best available test we can impose on a nuclear model used to predict the $\beta\beta_{0\nu}$ matrix elements.

Many experimental groups have reported measurements of $\beta\beta_{2\nu}$ processes [3]. Nearly for all the cases the ground-state (g.s.) to-ground-state ($0^+ \to 0^+$) decay was investigated. In direct-counting experiments the analysis of the sum-energy spectrum of the emitted electrons allowed the identification of the different $\beta\beta$-decay modes [4].

Only recently the possibility of detecting $\beta\beta$ decays into excited states of the daughter nucleus by measuring the gamma radiation has exerted some attraction among experimentalists. This was due to the fact that phase-space integrals scale as the energy available for the decay and consequently decrease for excited states. In the case of the decay to a first-excited 2^+ state the phase-space factor contains terms which are antisymmetric in the energies of the two outgoing electrons and antineutrinos, resulting in a large reduction of the corresponding integral. It makes such transitions very difficult to observe [5,6]. Also they are inclusive experiments which cannot distinguish between the different $\beta\beta$-decay modes. On the other hand the detection of a gamma ray gives a much clearer signal than a continuous electron spectrum as in the case of the g.s. $\to$ g.s. decay.

The pioneer work of Bellotti et al. [7] determined lower limits for the $\beta\beta$ decays to excited states of six nuclei. Lower limits for the decay of ^{128}Te and ^{130}Te to the first 2^+ state of the corresponding xenon isotopes have been reported [8]. The $\beta\beta$ decay of ^{76}Ge to excited states of ^{76}Se was studied looking for the detection of one or two photons in coincidence with two electrons [9,10]. The half-lives for the $\beta\beta_{2\nu}$ decay from the g.s. of ^{100}Mo, ^{96}Zr and ^{150}Nd to the first-excited 0^+ state of the daughter nuclei have been estimated for the first time in [11] assuming that the nuclear matrix elements of the transitions to the g.s. and excited 0^+ state are the same. Experimental studies of the decay to excited states looking for the γ-ray signature have been performed for ^{96}Zr [12], ^{116}Cd [4,13,14] and ^{100}Mo [13,15–17]. The detection of the $\beta\beta_{2\nu}$ to the first-excited 0^+ state was reported for the first time in [17,18]. The feasibility of studying the $\beta\beta_{2\nu}$ to excited states in ^{150}Nd has been discussed in recent years [4,11,12,19]. In [12] preliminary results were reported.

Theoretical analyses of the $\beta\beta_{2\nu}$ decays to excited states have been performed in the context of the QRPA formalism for ^{100}Mo [20,21], ^{136}Xe [22,23], ^{76}Ge [24,25], ^{116}Cd [4] and in [26] also for ^{82}Se, ^{110}Pd and ^{128,130}Te. When the first-excited 0^+ state was studied it was assumed that it is a member of a two-quadrupole–phonon triplet. The QRPA calculation for ^{100}Mo exhibits an overestimation of the amplitude of the $\beta\beta_{2\nu}$ decay to this excited state when the decay to the ground state is reproduced [20,21]. For the case of ^{116}Cd [4] the calculated matrix element of this decay is five times greater than that associated with the decay to the ground state.

Although there is no reported calculation of the $\beta\beta_{2\nu}$ decay of ^{150}Nd to excited states of ^{150}Sm, this nucleus was mentioned as a suitable candidate to study this decay. In [11] this conclusion has been reached assuming that the nuclear matrix elements for the $\beta\beta_{2\nu}$

decay to the ground state and the first excited 0^+ state are equal. In [4] it was speculated that if the matrix element for the $\beta\beta_{2\nu}$ decay of ^{150}Nd would show a similar enhancement over that of the g.s. decay as found for ^{116}Cd, the decay rate into this excited level could even exceed that of the g.s. decay.

In the present paper we perform an analysis of the $\beta\beta_{2\nu}$ decay of ^{150}Nd to excited 0^+ and 2^+ states of ^{150}Sm using the pseudo SU(3) formalism. In order to do this the necessary formalism to study the $\beta\beta_{2\nu}$ decay to 2^+ states has been developed. The pseudo SU(3) model is well suited to describe the collective spectra. Under this scheme the first 2^+ and 4^+ states are members of the g.s. rotational band, while the excited 0^+ states are the heads of excited rotational bands. We will show that within the pseudo SU(3) model the $\beta\beta_{2\nu}$ decay to the first-excited 0^+ state is cancelled while the other decays are strongly suppressed.

In Section 2 the pseudo SU(3) formalism and the model hamiltonian are briefly reviewed. In Section 3 the summation method is used to obtain the $\beta\beta_{2\nu}$ matrix elements for the decay to the 2^+ states. Section 4 contains the explicit formulae needed to evaluate the $\beta\beta_{2\nu}$ matrix elements in the pseudo SU(3) scheme. The nuclear-structure analysis of ^{150}Sm is given in Section 5. In Section 6 the $\beta\beta_{2\nu}$ nuclear matrix elements and half-lives are presented. Conclusions are drawn in the last section.

2. The pseudo SU(3) formalism

In order to obtain a microscopical description of the low-lying energy states of ^{150}Nd and ^{150}Sm we will use the pseudo SU(3) model which successfully describes collective excitations in rare-earth nuclei and actinides [27] as well as the g.s. $\rightarrow$ g.s. $\beta\beta_{2\nu}$ and $\beta\beta_{0\nu}$ decays of six heavy deformed nuclei [28–30].

In the pseudo SU(3) shell-model coupling scheme [31], normal-parity orbitals (η, l, j) are identified with orbitals of a harmonic oscillator of one quantum less, $\tilde{\eta} = \eta - 1$. This set of orbitals with $\tilde{j} = j = \tilde{l} + \tilde{s}$, pseudo spin $\tilde{s} = \frac{1}{2}$ and pseudo orbital angular momentum $\tilde{l}$ define the so-called pseudo space. Recently there was found an analytic expression for the transformation of the normal-parity orbitals to the pseudo space [32]. Applying this transformation to the spherical Nilsson hamiltonian it can be shown explicitly that the strength of the pseudo spin–orbit interaction is almost zero for heavy nuclei and the orbitals $j = \tilde{l} \pm \frac{1}{2}$ are nearly degenerate. For configurations of identical particles occupying a single j-orbital of abnormal parity a convenient characterization of states is made by means of the seniority coupling scheme.

The many-particle states of n_α nucleons in a given shell η_α, $\alpha = \nu$ or π, can be defined by the totally antisymmetric irreducible representations $\{1^{n_\alpha^{\rm N}}\}$ and $\{1^{n_\alpha^{\rm A}}\}$ of unitary groups. The dimensions of the normal-(N) parity space is $\Omega_\alpha^{\rm N} = (\tilde{\eta}_\alpha + 1)(\tilde{\eta}_\alpha + 2)$ and that of the unique (A) space is $\Omega_\alpha^{\rm A} = 2\eta_\alpha + 4$, with the constraint $n_\alpha = n_\alpha^{\rm A} + n_\alpha^{\rm N}$. Proton and neutron states are coupled to angular momentum $J^{\rm N}$ and $J^{\rm A}$ in both the normal- and unique-parity sectors, respectively. The wave function of the many-particle state with angular momentum J and projection M is expressed as a direct product of the normal- and unique-parity ones, as

$$|JM\rangle = \sum_{J^N,J^A} (|J^N> \otimes |J^A\rangle)^J_M. \tag{1}$$

We are interested in describing the low-lying energy states of ^{150}Nd and ^{150}Sm.

For even–even heavy nuclei it has been shown that if the residual neutron–proton interaction is of the quadrupole type, regardless of the interaction in the proton and neutron spaces, the most important normal-parity configurations are those with highest spatial symmetry $\{\tilde{f}_\alpha\} = \{2^{n^N_\alpha/2}\}$ [27]. This statement is valid for yrast states below the backbending region. This implies that $\tilde{S}_\pi = \tilde{S}_\nu = 0$, i.e. only pseudo spin-zero configurations are considered.

Additionally in the abnormal-parity space only seniority-zero configurations are taken into account. This simplification implies that $J^A_\pi = J^A_\nu = 0$. This is a very strong assumption, quite useful in order to simplify the calculations. Its effects upon the present calculation are discussed below.

The double-beta decay, when described in the pseudo SU(3) scheme, is strongly dependent on the occupation numbers for protons and neutrons in the normal- and abnormal-parity states n^N_π, n^N_ν, n^A_π, n^A_ν [28,29]. These numbers are determined filling the Nilsson levels from below, as discussed in [28,29]. In particular the $\beta\beta_{2\nu}$ decay is allowed only if it fulfils the following relationships:

$$\begin{aligned} &n^A_{\pi,f} = n^A_{\pi,i} + 2, \quad && n^A_{\nu,f} = n^A_{\nu,i}, \\ &n^N_{\pi,f} = n^N_{\pi,i}, && n^N_{\nu,f} = n^N_{\nu,i} - 2. \end{aligned} \tag{2}$$

A deformation $\beta \approx 0.28$ [33] was assumed for ^{150}Nd, and one of $\beta \approx 0.19$ [33] for ^{150}Sm. We were forced to select similar deformations for both nuclei to satisfy relations (2). If we assign the occupation numbers for these nuclei using the above-mentioned deformations, for each one the $\beta\beta_{2\nu}$ decay becomes forbidden. This result is strongly connected with the fact that configuration mixing (for example pairing effects) are not included in the present calculations, as will be discuss later.

We will focus on the analysis in the case $\beta \approx 0.28$. According to [34] this higher deformation is more appropriate for ^{152}Sm than for ^{150}Sm and is related with some departure from a rotational behavior in the ground-state band of ^{150}Sm. In this case we have obtained the occupation numbers (case I, $\beta \approx 0.28$)

$$\begin{aligned} &^{150}\text{Nd}: \quad n^A_\pi = 4, \quad n^N_\pi = 6, \quad n^A_\nu = 2, \quad n^N_\nu = 6; \\ &^{150}\text{Sm}: \quad n^A_\pi = 6, \quad n^N_\pi = 6, \quad n^A_\nu = 2, \quad n^N_\nu = 4. \end{aligned} \tag{3}$$

It is also possible to study both nuclei at a smaller deformation. This second case, which will be shortly mentioned below, has occupation numbers (case II, $\beta \approx 0.19$)

$$\begin{aligned} &^{150}\text{Nd}: \quad n^A_\pi = 2, \quad n^N_\pi = 8, \quad n^A_\nu = 0, \quad n^N_\nu = 8; \\ &^{150}\text{Sm}: \quad n^A_\pi = 4, \quad n^N_\pi = 8, \quad n^A_\nu = 0, \quad n^N_\nu = 6. \end{aligned} \tag{4}$$

In order to analyze the spectra and transition amplitudes of ^{150}Sm we have selected the standard version of the pseudo SU(3) hamiltonian [27]. It is constructed by a spherical central potential, a quadrupole–quadrupole interaction and a residual force. The latter allows the fine tuning of low-lying spectral features like K-band splitting and the effective moments of inertia. The hamiltonian looks like

$$H = \sum_{\alpha} H_{\alpha} - \tfrac{1}{2}\chi \boldsymbol{Q}^{\mathrm{a}} \cdot \boldsymbol{Q}^{\mathrm{a}} + \zeta_1 K^2 + \zeta_2 L^2. \tag{5}$$

The spherical Nilsson hamiltonian which describe the single-particle motion of neutrons ($\alpha = \nu$) or protons ($\alpha = \pi$) is

$$\begin{aligned} H_{\alpha} &= \sum_{s} \hbar\,\omega\left(\eta_{\alpha s} + \tfrac{3}{2} - 2k_{\alpha}\vec{L}_{\alpha s}\cdot\vec{S}_{\alpha s} - k_{\alpha}\,\mu_{\alpha} L^2_{\alpha s}\right) - V_{\alpha} \\ &= \sum_{s,\alpha} \epsilon_{s\alpha}\, a^{\dagger}_{s\alpha}\, a_{s\alpha}, \end{aligned} \tag{6}$$

where $\eta = \tilde{\eta} + 1$ denotes the harmonic-oscillator number operator and $\hbar\,\omega$ determines the size of the shell. A constant term V_{ν} (V_{π}) is included which represents the depth of the neutron (proton) potential well. In (6) the second-quantization representation of H_{α} is given, $\epsilon_{s\alpha}$ being the single-particle energies.

The quadrupole operator $\boldsymbol{Q}^{a} = \Sigma_{\mathrm{s}}(q_{\pi_{\mathrm{s}}} + q_{\nu_{\mathrm{s}}})$ acts only within a shell and does not mix different shells. The residual interaction, K^2, is a linear combination of L^2, X_3 and X_4, defined as

$$\begin{aligned} L^2 &= \sum_{i}^{3} L_i^2, \\ X_3 &= \sum_{i,j}^{3} L_i Q^a_{ij} L_j, \\ X_4 &= \sum_{i,j,k}^{3} L_i Q^a_{ij} Q^a_{jk} L_k. \end{aligned} \tag{7}$$

They are rotational-invariant operators built by generators of the algebra of SU(3) [27,29], and L_i and Q^{α}_{ij} are cartesian forms of the total angular momentum and the quadrupole operators, respectively. The K is interpreted to be the third component of the total angular momentum of an intrinsic body-fixed symmetry axis of the system, which is given by

$$K^2 = \left(\lambda_1\lambda_2 L^2 + \lambda_3 X_3 + X_4\right)/\left(2\lambda_3^2 + \lambda_1\lambda_2\right), \tag{8}$$

with the parameters λ_i denoting the eigenvalues of the mass quadrupole operator, which are related to the SU(3) labels (λ, μ) through the expressions

$$\lambda_1 = \tfrac{1}{3}(\mu - \lambda), \quad \lambda_2 = -\tfrac{1}{3}(\lambda + 2\mu + 3), \quad \lambda_3 = \tfrac{1}{3}(2\lambda + \mu + 3). \tag{9}$$

Although the quantum number K used to define the orthonormalized basis is not the same as the Elliott κ, the states studied in the present work satisfy quite accurately the relationships [35]

$$K^2\,|\,K=1\rangle = 0, \quad K^2\,|\,K=2\rangle \approx 4\,|\,K=2\rangle. \tag{10}$$

It would be possible to add to the hamiltonian (5) terms which distinguish different irreps. For the sake of simplicity we keep this simplest version.

With the occupation numbers determined in Eq. (3) and the hamiltonian (5) the wave function of the deformed ground state of ^{150}Nd can be written [28,29]

$$|{}^{150}\mathrm{Nd},0^+\rangle \equiv |0_i^+\rangle = |(\mathrm{h}_{11/2})^4_\pi,\ J_\pi^{\mathrm{A}} = M_\pi^{\mathrm{A}} = 0;\ (\mathrm{i}_{13/2})^2_\nu,\ J_\nu^{\mathrm{A}} = M_\nu^{\mathrm{A}} = 0\rangle_{\mathrm{A}}$$
$$|\{1^6\}_\pi\{2^3\}_\pi(12,0)_\pi;\ \{1^6\}_\nu\{2^3\}_\nu(18,0)_\nu;\ 1(30,0)\ K=1\ J=M=0\rangle_{\mathrm{N}}, \tag{11}$$

and the deformed low-energy states of ^{150}Sm are described by the wave functions

$$|{}^{150}\mathrm{Sm},J_\sigma^+\rangle \equiv |J_\sigma^+\rangle = |(\mathrm{h}_{11/2})^6_\pi,\ J_\pi^{\mathrm{A}} = M_\pi^{\mathrm{A}} = 0;\ (\mathrm{i}_{13/2})^2_\nu,$$
$$J_\nu^{\mathrm{A}} = M_\nu^{\mathrm{A}} = 0\rangle_{\mathrm{A}}\,|\{1^6\}_\pi\{2^3\}_\pi(12,0)_\pi;\ \{1^4\}_\nu\{2^2\}_\nu(12,2)_\nu;$$
$$1(\lambda,\mu)_\sigma K J M\rangle_{\mathrm{N}}, \tag{12}$$

where J_σ^+ denotes a state with angular momentum J, positive parity and associated with the SU(3) irrep $(\lambda, \mu)_\sigma$. In this approach we are assuming that the first 0^+, 2^+, 4^+ states of ^{150}Sm are the low-energy sector of a rotational band described by the normal $(\lambda, \mu)_{\mathrm{g.s.}} = (24, 2)$ strong-coupled pseudo SU(3) irrep, the second 0_2^+ and third 2_3^+ states belong to a second rotational band with $(\lambda, \mu)_1 = (20, 4)$, and the third 0_3^+ state is the head of another rotational band described by the pseudo SU(3) irrep $(\lambda, \mu)_2 = (22, 0)$. We will discuss also a gamma band associated with $(\lambda, \mu)_{\mathrm{g.s.}}$ with $K = 2$.

3. The $\beta\beta_{2\nu}$ decay to excited 2^+ states

The inverse half-life of the two-neutrino mode of the $\beta\beta$ decay can be cast in the form [5]

$$\left[\tau_{2\nu}^{1/2}(0^+ \to J_\sigma^+)\right]^{-1} = G_{2\nu}(J_\sigma^+)\,|M_{2\nu}(J_\sigma^+)|^2, \tag{13}$$

where $G_{2\nu}(J_\sigma^+)$ are kinematical factors. They depend on $E_{J,\sigma} = \frac{1}{2}[Q_{\beta\beta} - E(J_\sigma)] + m_e c^2$ which is half of the total energy released. The nuclear matrix element is

$$M_{2\nu}(J_\sigma^+) \approx M_{2\nu}^{\mathrm{GT}}(J_\sigma^+) = \frac{1}{\sqrt{J+1}} \sum_N \frac{\langle J_\sigma^+ \| \Gamma \| 1_N^+\rangle\langle 1_N^+ \| \Gamma \| 0_i^+\rangle}{\mu_N^{J+1}}, \tag{14}$$

with the Gamow–Teller operator Γ expressed as

$$\Gamma_m = \sum_s \sigma_{ms} t_s^- \equiv \sum_{\pi\nu} \sigma(\pi,\nu)\left(a^\dagger_{\eta_\pi l_\pi \frac{1}{2}; j_\pi} \otimes \tilde{a}_{\eta_\nu l_\nu \frac{1}{2}; j_\nu}\right)^1_m, \quad m = 1, 0, -1. \tag{15}$$

The energy denominator is $\mu_N = E_{J,\sigma} + E_N - E_i$ and it contains the intermediate E_N and initial E_i energies. The kets $|1_N^+\rangle$ denote intermediate states.

The mathematical expressions needed to evaluate the nuclear matrix elements of the g.s. → g.s. $\beta\beta_{2\nu}$ decay in the pseudo SU(3) model were developed recently [28,29]. The same formulae describe the decay to the first-excited 0^+ state by replacing the values of the strong-coupled irrep $(\lambda, \mu)_{\mathrm{g.s.}}$ of Eq. (12) with those corresponding to excited bands.

We will concentrate first on the derivation of the matrix element $M_{2\nu}(2^+_\sigma)$ to the 2^+ excited states. The formulae for this decay resembles that of the decay to the 0^+ states but the energy denominator is up to the third power. This energy being in general of the order of 10 MeV, this power implies a factor 100 of suppression for this matrix element [5,20,21]. The previous equation is rearranged as

$$M^{\mathrm{GT}}_{2\nu}(2^+_\sigma) = \sqrt{5} \sum_{\mu\mu'} (1\mu 1\mu' | 2m) \sum_{Nm_1} \mu_N^{-3} \langle 2^+_\sigma m | \Gamma_{\mu'} | 1^+_N m_1 \rangle \langle 1^+_N m_1 | \Gamma_\mu | 0^+_i \rangle . \tag{16}$$

Using

$$\mu_N^{-3} = \frac{1}{2} \frac{\partial^2}{\partial E^2_{J,\sigma}} \mu_N^{-1} \tag{17}$$

and the summation method described in [29,36] it is possible to rewrite the second sum as

$$\sum_{Nm_1} \frac{1}{2} \frac{\partial^2}{\partial E^2_{J,\sigma}} \left(\mu_N^{-1} \langle 2^+_\sigma m | \Gamma_{\mu'} | 1^+_N m_1 \rangle \langle 1^+_N m_1 | \Gamma_\mu | 0^+_i \rangle \right)$$

$$= \frac{1}{2} \frac{\partial^2}{\partial E^2_{J,\sigma}} \langle 2^+_\sigma m | \sum_{\lambda=1}^{\infty} \frac{(-1)^\lambda}{E^\lambda_{J,\sigma}} \Gamma_{\mu'} \left[H, \left[H, \ldots, \left[H, \Gamma_\mu \right] \ldots \right]^{(\lambda\text{-times})} | 0^+_i \rangle . \tag{18}$$

The two-body terms of the hamiltonian (5) commute with the Gamow–Teller operator (15), thus the above multiple commutators are easy to evaluate. We obtain [29]

$$\begin{aligned} \left[H, \ldots \left[H, \Gamma_\mu \right] \right] \ldots \big]^{(\lambda\text{-times})} &= \left[H_\pi + H_\nu \ldots \left[H_\pi + H_\nu, \Gamma_\mu \right] \right] \ldots \big]^{(\lambda\text{-times})} \\ &= \sum_{\pi\nu} \sigma(\pi, \nu) \left(a^\dagger_\pi \otimes \tilde{a}_\nu \right)^{1\mu} (\epsilon_\pi - \epsilon_\nu)^\lambda , \end{aligned} \tag{19}$$

where $\pi \equiv (\eta_\pi, l_\pi, j_\pi)$ and $\nu \equiv (\eta_\nu, l_\nu, j_\nu)$. Returning with this expression to the original formula, Eq. (13), resumming the infinite series and recoupling the Gamow–Teller operators, it is found that

$$\begin{aligned} M^{\mathrm{GT}}_{2\nu}(2^+_\sigma) &= \sqrt{5} \frac{1}{2} \frac{\partial^2}{\partial E^2_{J,\sigma}} \Bigg(\sum_{\pi\nu, \pi'\nu'} \frac{\sigma(\pi, \nu) \sigma(\pi', \nu')}{E_{J,\sigma} + \epsilon_\pi - \epsilon_\nu} \\ &\qquad \times \langle 2^+_\sigma m | \left[\left(a^\dagger_\pi \otimes \tilde{a}_\nu \right)^1 \otimes \left(a^\dagger_{\pi'} \otimes \tilde{a}_{\nu'} \right)^1 \right]^{2m} | 0^+_i \rangle \Bigg) \\ &= \sqrt{5} \sum_{\pi\nu, \pi'\nu'} \frac{\sigma(\pi, \nu) \sigma(\pi', \nu')}{(E_{J,\sigma} + \epsilon_\pi - \epsilon_\nu)^3} \\ &\qquad \times \langle 2^+_\sigma m | \left[\left(a^\dagger_\pi \otimes \tilde{a}_\nu \right)^1 \otimes \left(a^\dagger_{\pi'} \otimes \tilde{a}_{\nu'} \right)^1 \right]^{2m} | 0^+_i \rangle . \end{aligned} \tag{20}$$

As it was shown in [29] and mentioned above the expression for the nuclear matrix element of the g.s. → g.s. $\beta\beta_{2\nu}$ decay is similar to (20) with a different power in the denominator. Eqs. (20) and (4.10) of [29] can be expressed in a compact form as follows:

$$M_{2\nu}^{\mathrm{GT}}(J_\sigma^+) = \sqrt{J+3}\sum_{\pi\nu,\pi'\nu'} \frac{\sigma(\pi,\nu)\sigma(\pi',\nu')}{(E_{J,\sigma}+\epsilon_\pi-\epsilon_\nu)^{J+1}}$$
$$\times\langle J_\sigma^+ m|\left[\left(a_\pi^\dagger\otimes\tilde{a}_\nu\right)^1\otimes\left(a_{\pi'}^\dagger\otimes\tilde{a}_{\nu'}\right)^1\right]^{Jm}|0_i^+\rangle$$
$$\equiv\sum_{\pi\nu,\pi'\nu'}\frac{1}{(E_{J,\sigma}+\epsilon_\pi-\epsilon_\nu)^{J+1}}\langle J_\sigma^+ m|T^{Jm}(\pi\nu,\pi'\nu')|0_i^+\rangle. \tag{21}$$

For practical purposes the tensor $T^{Jm}(\pi\nu,\pi'\nu')$ was implicitly defined in the above equation. The $\sqrt{3}$ for the $J=0$ case comes from the relation between the scalar product of two vectors and their coupling to angular momentum zero.

4. The matrix elements $M_{2\nu}$

We want to evaluate the nuclear matrix element (21) for the $\beta\beta_{2\nu}$ decay of the ground state of ^{150}Nd, Eq. (11), to the ground and excited states of ^{150}Sm which are described by the wave functions of Eq. (12). Each Gamow–Teller operator (15) annihilates a proton and creates a neutron in the same oscillator shell and with the same orbital angular momentum. In the case of the $\beta\beta_{2\nu}$ decay of ^{150}Nd it means that the operator annihilates two neutrons in the pseudo shell $\eta_\nu=5$ and creates two protons in the abnormal orbit $h_{11/2}$. As a consequence the only orbitals which in the model space can be connected through the $\beta\beta_{2\nu}$ decay are those satisfying $\eta_\pi=\eta_\nu\equiv\eta$, that implies $l_\pi=l_\nu=\eta, j_\nu=\eta-\frac{1}{2}$ and $j_\pi=\eta+\frac{1}{2}$. These are the selection rules described by relations (2) concerning the change in occupation numbers. Under these restrictions only one term in the sum over configurations $\pi\nu,\pi'\nu'$ survives and thus the nuclear matrix element $M_{2\nu}$ (21) can be written as

$$M_{2\nu}^{\mathrm{GT}}(J_\sigma^+) = \frac{1}{\mathscr{E}_{J\sigma}^{J+1}}\langle J_\sigma^+ |T^{Jm}(\pi\nu,\pi\nu)|0_i^+\rangle, \tag{22}$$

where the energy denominator is determined demanding that the isobaric analog state in the intermediate odd–odd nucleus is an eigenstate of the hamiltonian (5). Its excitation energy is equal to the difference in Coulomb energies Δ_C. Their expressions are [29]

$$\mathscr{E}_{J\sigma}=E_{J\sigma}+\epsilon(\eta_\pi,l_\pi,j_\pi=j_\nu+1)-\epsilon(\eta_\nu,l_\nu,j_\nu)=E_{J\sigma}-\hbar\omega k_\pi 2j_\pi+\Delta_\mathrm{C},$$
$$\Delta_\mathrm{C}=\frac{0.70}{A^{1/3}}\left[2Z+1-0.76\left[(Z+1)^{4/3}-Z^{4/3}\right]\right]\ \mathrm{MeV}. \tag{23}$$

As it was discussed in [29] in the context of the g.s. → g.s. $\beta\beta_{2\nu}$ decay, Eq. (22) has no free parameters, the denominator (23) being a well-defined quantity. The reduction to only one term comes as a consequence of the restricted Hilbert proton and neutron spaces of the model. The initial and final ground states are strongly correlated with a very rich structure in terms of their shell-model components.

Since the Hilbert space has been divided in their normal and unique-parity components we need to rearrange the creation and annihilation operators in the same way, i.e.

$$\left[\left(a_\pi^+ \otimes \tilde{a}_\nu\right)^1 \otimes \left(a_\pi^\dagger \otimes \tilde{a}_\nu\right)^1\right]^{JM}$$

$$= \sum_{J_\pi J_\nu} \chi\begin{Bmatrix} j_\pi & j_\nu & 1 \\ j_\pi & j_\nu & 1 \\ J_\pi & J_\nu & J \end{Bmatrix} \left[\left(a_\pi^\dagger \otimes a_\pi^\dagger\right)^{J_\pi} \otimes \left(\tilde{a}_\nu \otimes \tilde{a}_\nu\right)^{J_\nu}\right]^{JM}. \qquad (24)$$

The $\chi(\ldots)$ is the unitary (Jahn–Hope) 9-j recoupling coefficient [37]. Introducing this expression in (22) together with the explicit form of the wave functions (11) and (12), we obtain

$$M_{2\nu}^{\mathrm{GT}}\left(J_\sigma^+\right)$$

$$= \sqrt{J+3}\ \sigma(\pi,\nu)^2 \mathscr{E}_{J\sigma}^{-(J+1)} \sum_{J_\pi J_\nu} \chi\begin{Bmatrix} j_\pi & j_\nu & 1 \\ j_\pi & j_\nu & 1 \\ J_\pi & J_\nu & J \end{Bmatrix}$$

$$\times\Big[\langle\left(\mathrm{h}_{11/2}\right)_\pi^6, J_\pi^{\mathrm{A}} = M_\pi^{\mathrm{A}} = 0 \,|\left(a_\pi^\dagger \otimes a_\pi^\dagger\right)^{J_\pi}|\left(\mathrm{h}_{11/2}\right)_\pi^4, J_\pi^{\mathrm{A}} = M_\pi^{\mathrm{A}} = 0\rangle$$

$$\otimes\langle(12,0)_\pi;(12,2)_\nu;1(\lambda,\mu)_\sigma\ K=1\ J\,M\,|\left(\tilde{a}_\nu \otimes \tilde{a}_\nu\right)^{J_\nu}$$

$$\times|(12,0)_\pi;(18,0)_\nu;1(30,0)\ K=1\ J=M=0\rangle\Big]^{JM}. \qquad (25)$$

The matrix element (25) vanishes unless a pair of protons coupled to total angular momentum zero is created and two neutrons of the normal-parity space coupled to pseudo orbital angular momentum $\tilde{L} = J$ and pseudo spin equal to zero are annihilated. The above sum is thus restricted to $J_\pi = 0$, $J_\nu = J$.

The operators in the normal space must be recoupled from the jj- to the LS-coupling scheme. The result is

$$\left(\tilde{a}_\nu \otimes \tilde{a}_\nu\right)^{JM} = \sum_{\tilde{L}\tilde{S}} \chi\begin{Bmatrix} \tilde{l}_\nu & \frac{1}{2} & j_\nu \\ \tilde{l}_\nu & \frac{1}{2} & j_\nu \\ \tilde{L} & \tilde{S} & J \end{Bmatrix} \left(\tilde{a}_{(0\tilde{\eta})\tilde{l}\frac{1}{2}} \otimes \tilde{a}_{(0\tilde{\eta})\tilde{l}\frac{1}{2}}\right)_{JM}^{\tilde{L}\tilde{S}}. \qquad (26)$$

The low-energy levels are assumed to have pseudo spin $\tilde{S} = 0$, a fact that again simplifies the evaluation of the above sum by imposing $\tilde{L} = J$.

Using $j_\pi = j_\nu + 1, l_\pi = l_\nu = \eta$, the reduced matrix elements $\sigma(\pi,\nu)$ read

$$\sigma(\pi,\nu)^2 = \frac{8\eta(\eta+1)}{3(2\eta+1)}. \qquad (27)$$

In the seniority-zero approximation the two-particle transfer matrix element of the unique-parity sector of Eq. (22) is evaluated by using the quasi-spin formalism and gives [28,29]

$$\langle j_\pi^{n_\pi^A+2}, J_\pi^A = M_\pi^A = 0 \,|\, \left[a_\pi^\dagger \otimes a_\pi^\dagger\right]^0 \,|\, j_\pi^{n_\pi^A}, J_\pi^A = M_\pi^A = 0\rangle$$
$$= \left(\frac{\left(n_\pi^A+2\right)\left(\eta+1-\frac{1}{2}n_\pi^A\right)}{\eta+1} \right)^{1/2}. \qquad (28)$$

The evaluation of the matrix elements in the normal space of Eq. (25) is performed by using SU(3) Racah calculus to decouple the proton and neutron normal irreps, and expanding the annihilation operators of Eq. (26) in their SU(3) tensorial components. The final result is

$$M_{2\nu}^{\mathrm{GT}}(J_\sigma^+) = a(J)\, b(n_\pi^A)\, \mathscr{E}_{J\sigma}^{-(J+1)}$$
$$\times \sum_{(\lambda_0\mu_0)K_0} \langle (0\tilde{\eta})\ 1\ \tilde{l},\ (0\tilde{\eta})\ 1\ l \,\|\, (\lambda_0\mu_0)\ K_0\ J\rangle_1$$
$$\times \sum_{\rho} \langle (30,0)\ 1\ 0,\ (\lambda_0\mu_0)\ K_0\ J \,\|\, (\lambda\mu)_\sigma\ 1\ J\rangle_\rho$$
$$\times \sum_{\rho'} \begin{pmatrix} (12,0) & (0,0) & (12,0) & 1 \\ (18,0) & (\lambda_0\mu_0) & (12,2) & \rho' \\ (30,0) & (\lambda_0\mu_0) & (\lambda\mu)_\sigma & \rho \\ 1 & 1 & 1 & \end{pmatrix}$$
$$\times \langle (12,2) \,|||\, \left(\tilde{a}_{(0\tilde{\eta})\frac{1}{2}} \tilde{a}_{(0\tilde{\eta})\frac{1}{2}}\right)^{(\lambda_0\mu_0)} \,|||\, (18,0)\rangle_{\rho'}. \qquad (29)$$

In the above formula $\langle \ldots, \ldots \|.\rangle$ denotes the SU(3) Clebsch–Gordan coefficients [38], the symbol $[\ldots]$ represents a $9-\lambda\mu$ recoupling coefficient [39], and $\langle \ldots \,|||\, \ldots \,|||\, \ldots\rangle$ denotes the triple-reduced matrix elements[40]. The energy denominator was defined in (23), and

$$a(0) = \frac{4\eta}{(2\eta+1)\sqrt{2\eta-1}}, \quad a(2) = \frac{2}{2\eta+1}\left(\frac{5\eta(\eta-1)(2\eta-3)}{3(2\eta+1)}\right)^{1/2},$$
$$b(n_\pi^A) = \left[\left(n_\pi^A+2\right)\left(\eta+1-\tfrac{1}{2}n_\pi^A\right)\right]^{1/2}. \qquad (30)$$

5. The rotational spectrum of ^{150}Sm

The three parameters of the pseudo SU(3) hamiltonian (5) were fitted to reproduce the first 2^+ states in ^{150}Sm as it was done in [27]. Their values are

$$\chi = 3.47 \text{ eV}, \quad \zeta_1 = 215 \text{ keV}, \quad \zeta_2 = 50.4 \text{ keV}. \qquad (31)$$

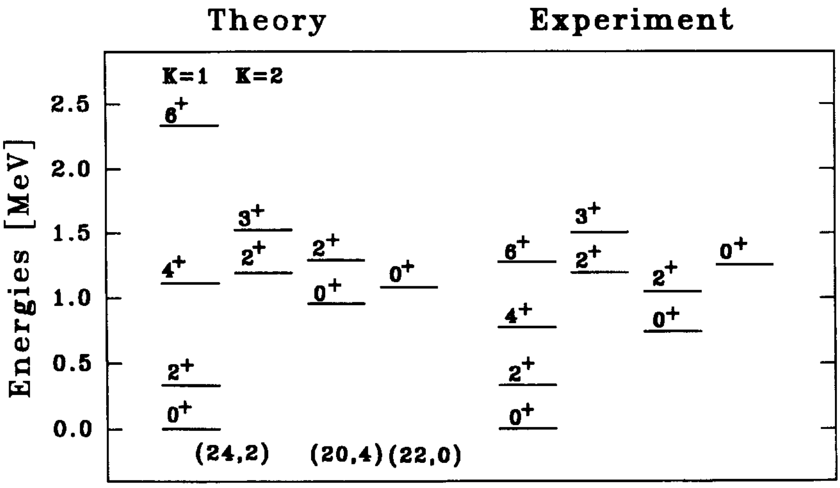

Fig. 1. Spectrum of the low-lying states of ^{150}Sm. The levels are grouped in rotational bands and they are labeled by angular momentum and parity. The right-hand side contains the experimentally determined levels. In the left-hand side the calculated spectrum is exhibited with the associated irreps at the bottom.

The right-hand side of Fig. 1 exhibits nine of the lowest energy states which have been observed in ^{150}Sm, grouped in rotational bands. Angular momentum and parity are given for each level. The left-hand side of Fig. 1 shows the calculated spectrum together with the associated irreps. The gamma band is identified with $K = 2$. The general trend is well reproduced but the experimental g.s. band does not show the rotational structure which is exhibited by the calculated one. This departure from an exact rotational behavior was mentioned in Section 2 and it can be associated with the relatively small deformation reported for ^{150}Sm. In this mass region the deformation suddenly jumps for ^{152}Sm to a value very similar to the deformation of ^{150}Nd. The gamma band is present in the g.s. irrep because both λ and μ are different from zero, and is well fitted.

The excited 0^+ states are the head of other rotational bands. The predicted energy gap between them is 125 eV while the experimental one is 454 eV. These numbers suggest that we do not have a clear identification of these excited states. This fact has relevance in the study of the $\beta\beta_{2\nu}$ decay to these states.

The BE(2) transition intensities were evaluated using the effective quadrupole operator [27]

$$Q_0 = e_\pi^{\text{eff}} Q_\pi + e_\nu^{\text{eff}} Q_\nu, \quad e_\pi^{\text{eff}} = e + e_{\text{pol}}, \quad e_\nu^{\text{eff}} = e_{\text{pol}}, \tag{32}$$

with $e_{\text{pol}} = 0.93e$. The seniority-zero condition imposed on the nucleons in abnormal-parity orbitals inhibits them to participate in collective excitations. This restriction forces a slightly large value for the polarization charge. A similar effect was found in a BE(2) study of rare-earth and actinide nuclei [27].

The BE(2) intensities of the transition from the first and second 2^+ to the ground state, and from the first 4^+ to the 2^+ state are shown in Table 1 and compared with their experimental values, in Weisskopf units (W.u.). The agreement is good except for the

Table 1
B(E2) transition intensities

Transition	BE(2) intensity [W.u.]	
	Theory	Experiment
$2_1^+ \rightarrow 0_{\text{g.s.}}^+$	55.7	55.8
$2_2^+ \rightarrow 0_{\text{g.s.}}^+$	0.48	2.0
$4_1^+ \rightarrow 2_1^+$	78.5	112.0

case of the transition from the second-excited 2^+ state to the ground state which fails by a factor four.

6. The $\beta\beta_{2\nu}$ decay of ^{150}Nd

In this section we study the two-neutrino mode of the double-beta decay $\beta\beta_{2\nu}$ of ^{150}Nd into the ground state, the first-excited 2^+ and the first- and second-excited 0^+ states of ^{150}Sm.

In Table 2 the energy denominators, phase-space integrals, matrix elements and predicted half-lives for the $\beta\beta_{2\nu}$ decay of ^{150}Nd to the ground state, the first 2^+ and the first- and second-excited 0^+ states of ^{150}Sm are presented. The matrix elements (29) are given in units of $(m_e c^2)^{-(J+1)}$. The phase-space integrals for the decay to the 0^+ states were evaluated following the prescriptions given in [41] with $g_A/g_V = 1.0$, and the kinematical factor for the decay to the 2^+ state was taken from [5] renormalized by the above-mentioned value of the axial-vector coupling constant. It must be mentioned that these phase-space factors differ by about 10% with those used in [28,29] where a different renormalization procedure was used. For each decay the matrix elements and $\beta\beta_{2\nu}$ half-lives shown of Table 2 obtained with the case I (case II) occupation numbers are given in the upper (lower) row. Case I refers to Eq. (3) with $\beta \approx 0.28$ while case II to Eq. (4) with $\beta \approx 0.19$.

Table 2
Energy denominators, phase-space integrals, matrix elements and predicted half-lives for several transitions

	$\mathscr{E}_{J\sigma}$ [MeV]	$G_{2\nu}(J_\sigma^+)$ [yr^{-1}]	Case	$M_{2\nu}^{\text{GT}}(J_\sigma^+)$	$\tau_{2\nu}^{1/2}(0^+ \rightarrow J_\sigma^+)$ [yr]
$0^+ \rightarrow 0^+$ (g.s.)	12.20	4.94×10^{-17}	I	0.0549	6.73×10^{18}
			II	-0.0550	6.68×10^{18}
$0^+ \rightarrow 0^+$ (1)	11.83	5.83×10^{-18}	I	0	∞
			II	-0.00447	8.57×10^{21}
$0^+ \rightarrow 0^+$ (2)	11.58	9.33×10^{-19}	I	-0.00499	4.31×10^{22}
			II	0	∞
$0^+ \rightarrow 2^+$	12.04	4.78×10^{-17}	I	-5.38×10^{-5}	7.21×10^{24}
			II	4.12×10^{-5}	1.23×10^{25}

As it was mentioned in [29] the predicted half-life for the $\beta\beta_{2\nu}$ decay to the ground state of ^{150}Sm is in reasonable agreement with the experimental data, which vary between 9 and 17×10^{18} yr [3,42,43]. For case II these comments remain valid.

The $\beta\beta_{2\nu}$ decay to the first excited 0^+ state is *forbidden*. In this model this is imposed by the fulfillment of an exact selection rule. It can be understood by realizing that the pair of annihilation operators $\tilde{a}_{(0\bar{4})\frac{1}{2}}$, when expanded in their SU(3) components, have the couplings $(0, 4) \otimes (0, 4) = (0, 8),(2, 4)$ containing a $L = 0$ state. But acting over the ^{150}Nd g.s. irrep (30, 0) they cannot couple to the irrep (20, 4) which we associated with the first-excited 0^+ state. In other words the transition described by this pair of annihilation operators is forbidden between members of these particular irreps.

The decay to the second-excited 0^+ state is allowed but strongly cancelled. The reduction of the matrix element by a factor ten, in comparison with that associated with the decay to the g.s., is partly related with the fact that though the coupling $(30, 0) \otimes (0, 8) = (22, 0)$ is allowed the coupling with (2, 4) is forbidden. The predicted half-life is four orders of magnitude larger than that of the decay to the g.s..

The $\beta\beta_{2\nu}$ decay to the 2^+ state is inhibited by the μ_N^3 dependence of the matrix element as it is discussed in Section 3. The matrix element of the $\beta\beta_{2\nu}$ decay to the first-excited 2^+ state $M_{2\nu}(2^+_{\text{g.s.}})$ is three orders of magnitude less than the matrix element of the decay to the g.s. $M_{2\nu}(0^+_{\text{g.s.}})$. The present results differ from those previously published [4,11] where it was speculated that the $\beta\beta_{2\nu}$ decay of ^{150}Nd to the first-excited 0^+ state of ^{150}Sm could have a similar intensity as that of the g.s.. We find that in the present formalism this decay is forbidden.

If we select case II occupation numbers for both ^{150}Nd and ^{150}Sm, taken the deformation of the latter nucleus instead of that of the former, we find very similar results for the decay to the g.s. and the first 2^+ state, but the matrix elements of the decay to the first and second 0^+ states becomes interchanged with essentially the same values, as can be seen in columns 3 and 4 of Table 2, in the numbers given in the lower rows. Given the difference in the phase-space integrals we predict a half-life of the order of 10^{21} yr for the decay to the first-excited 0^+ state while the decay to the second-excited one becomes forbidden.

The above-discussed reduction of the matrix element of the $\beta\beta_{2\nu}$ decay to the excited 0^+ state as compared with the decay to the g.s. is not a general result of the pseudo SU(3) scheme. A recent analysis of the case of ^{100}Mo [44] shows that both matrix elements are very similar and that they are in agreement with the experimental information. In conclusion, the appearance of selection rules which can produce the suppression of the matrix elements governing a $\beta\beta_{2\nu}$ transition is a consequence of the details of the irreps involved.

The pseudo SU(3) model uses a quite restrictive Hilbert space. The model could be improved by incorporating mixing between different irreps, via pairing for example [45]. Also other active shells can be taken into account in the symplectic extension [46]. In both cases the selection rules that impose such strong restrictions on the $\beta\beta_{2\nu}$ decays of some nuclei can be superseded. However, if the main part of the wave function is well represented by the pseudo SU(3) model those forbidden decays will have, in the better case, matrix elements that will be no greater than 20% of the allowed ones, resulting in at least one order of magnitude cancellation in the half-life. In any case these results should be taken into account in the design of future experiments.

7. Conclusions

In the present paper we have studied the $\beta\beta_{2\nu}$-decay mode of ^{150}Nd to the ground and excited states of ^{150}Sm. The transitions have been analyzed in the context of the pseudo SU(3) model. The experimental spectrum of the g.s. rotational band of ^{150}Sm was reproduced as well as the measured half-life of the $\beta\beta_{2\nu}$ decay to the g.s., but the excited rotational bands were not so well reproduced. The $\beta\beta_{2\nu}$ decay to the first-excited 0^+ state was found forbidden in the model and the decay to the second-excited 0^+ state has a half-life four orders of magnitude greater that that of the g.s.. The decay to the 2^+ state is strongly inhibited due to the energy dependence of the matrix element $M_{2\nu}(2^+)$, two powers greater than that of the matrix element $M_{2\nu}(0^+)$.

It is expected that improving the model would remove the exact selection rules which forbid some decays. In any case, if the pseudo SU(3) wave functions are a good representation of the low-lying energy states of ^{150}Nd and ^{150}Sm, they will remain inhibited.

Acknowledgements

Work supported in part by CONACYT under contract 3513-E9310, and by a CONACYT-CONICET agreement under the project Double Beta Decay. O.C. is a fellow of the CONICET, Argentina.

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