Quantum computing in arrays coupled by “always-on” interactions

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It has recently been shown that one can perform quantum computation in a Heisenberg chain in which the interactions are always on, provided that one can abruptly tune the Zeeman energies of the individual (pseudo)spins. Here we provide a more complete analysis of this scheme, including several generalizations. We generalize the interaction to an anisotropic form (incorporating the $XY$ interaction as a limit), providing a proof that a chain coupled in this fashion tends to an effective Ising chain in the limit of far off-resonant spins. We derive the primitive two-qubit gate that results from exploiting abrupt Zeeman tuning with such an interaction. We also demonstrate, via numerical simulation, that the same basic scheme functions in the case of smoothly shifted Zeeman energies. We conclude with some remarks regarding generalizations to two- and three-dimensional arrays.

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There has recently been considerable interest in the question of whether one can perform quantum computation (QC) in Heisenberg-type systems (e.g., interacting electron spins) when the interaction is always on [1–3]. This question follows on from earlier work concerning Heisenberg systems in which the interactions are presumed to be switchable, either individually [4–6] or collectively [7]. Numerous proposals exist [10–12] for experimental realization of such a model, however interaction switching is liable to prove very challenging to achieve, and this motivates the interest in “always-on” interactions. In Ref. 1 we proposed a scheme for exploiting a simple one-dimensional Heisenberg chain with constant, isotropic nearest neighbor interactions. The scheme involved adjusting the single-spin level splittings (the Zeeman energies) to bring neighbors in and out of resonance with one another. We exploited the fact that far off-resonance spins do not exchange energy, but rather interact in an Ising $ZZ$ form. We argued that by separating the qubit-bearing spins by passive barrier spins, one can negate this residual interaction (thus achieving a passive state for the array)—yet one can invoke an interaction on demand simply by bringing a barrier into resonance with its neighbors.

In the present paper we elaborate on several aspects of Ref. [1], and we provide certain extensions. Whereas previously we considered only one specific form for the interaction, i.e., the isotropic Heisenberg form $(\sigma^x\sigma^x+\sigma^y\sigma^y+\sigma^z\sigma^z)$, we now generalize our arguments to accommodate different magnitudes for the in-plane and perpendicular components. Thus we subsume the prior isotropic form, and the purely planar $XY$ interaction, as special cases. There is a wide variety of promising physical systems associated with this family of interactions (for the isotropic limit, see, e.g., Refs. [10–12], and for the anisotropic case, Refs. [13–15]). Note that the $XY$ limit is also relevant to Förster-Dexter processes, e.g., in the context of excitonic exchange in biological molecules [8,9].

I. ANALYSIS OF HEISENBERG CHAIN WITH LARGE ZEEMAN DISCREPANCIES

The analysis is presented in full in Appendix A. Here we summarize it. We start from a total Hamiltonian $H$ given by

$$H = H_{\text{single}} + H_{\text{int}},$$

where

$$H_{\text{single}} = \sum_j B_j \sigma_j^z,$$

and the exchange interaction is as follows, where the factor $J_{XY}$ and $J_Z$ are distinguished to allow for a possible anisotropy between the in-plane and $z$-direction components,

$$H_{\text{int}} = \sum_j J_{XY}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + J_Z\sigma_j^z\sigma_{j+1}^z = \frac{J_{XY}}{2} \sum_j (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + J_Z\sum_j \sigma_j^z\sigma_{j+1}^z,$$

where $\sigma^s = \sigma^x \pm i\sigma^y$. Here and below, the sum ranges over all $N$ qubits, but subscripts such as $j+1$ are understood to be modulo $N$, i.e., we assume a closed circular topology. This

$$\frac{J_{XY}}{2} \sum_j (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + J_Z\sum_j \sigma_j^z\sigma_{j+1}^z,$$

where $\sigma^s = \sigma^x \pm i\sigma^y$. Here and below, the sum ranges over all $N$ qubits, but subscripts such as $j+1$ are understood to be modulo $N$, i.e., we assume a closed circular topology. This
considerably simplifies the analysis, but it is not a real constraint—in the limit of large chains the open and closed topologies will be equivalent.

We rewrite \( H = H_1 + H_2 \) where

\[
H_1 = \sum_j B_j \sigma_j^z + J_z \sum_j \sigma_j^z \sigma_{j+1}^z
\]

and

\[
H_2 = \frac{J_{XY}}{2} \left( \sum_j \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right).
\]

Notice that \( H_1 \) is simply the Hamiltonian for an Ising spin chain with varying Zeeman energies. We will find that this term dominates the time evolution when the spins are far off-resonance with their neighbors; the contribution of \( H_2 \) then vanishes.

Our approach is to exploit the Trotter formula to manipulate the time evolution operator into a form that can be recognized as Ising and non-Ising parts. This is detailed in Appendix A. The exact expression for the time evolution is found to be

\[
U(t) = R(t) \exp(-i H_1 t)
\]

where

\[
R = \prod_{n=1}^{\text{magnetic}} \exp \left[ -\frac{it}{n} H_R \left( \frac{mt}{n} \right) \right] \rightarrow _{n \rightarrow \infty}
\]

\[
H_R(\eta) = \frac{J_{XY}}{2} \sum_j X_j(\eta) \sigma_j^x \sigma_{j+1}^x + X_j^f(\eta) \sigma_j^y \sigma_{j+1}^y
\]

with

\[
X_j(\eta) = \exp \left[ i \eta (\Delta_j + 2J_Z (\sigma_j^z - \sigma_{j-1}^z)) \right],
\]

where \( \Delta_j = 2(B_{j+1} - B_j) \) and we use units such that \( \hbar = 1 \). The right-hand term in \( U(t) \) is the pure Ising chain evolution we seek, but the subsequent residual operator \( R \) is more complex. In the second part of Appendix A we expand \( R \) as a power series and inspect the terms. We conclude that, for a regular chain with a characteristic \( \Delta \) [such as an ABABAB- chain where \( \Delta_j = (-1)^j \Delta \)], the time evolution can be written as

\[
U(t) = [1 - \delta P(t)] \exp(-i H_1 t),
\]

where \( \delta = J_{XY} / \Delta \), for some finite operator \( P(t) \) whose magnitude does not increase with \( \Delta \). Thus for any given time period \( t \) the non-Ising evolution will be negligible if \( \Delta \) is sufficiently large compared to \( J_{XY} \). Assuming that we can dynamically change a \( \Delta \), switching it between zero and a large value, we can then exploit this result to produce a form of gate for quantum computation.

II. Exploitation of the Heisenberg-to-Ising Transition to Perform QC

Assume that we have some array in which every pair of adjacent spins is far off resonance from one another, i.e., \( \Delta \gg J_{XY} \), \( \forall j \). Now assume that we abruptly tune one (or more) of the spin Zeeman energies so that we have a triplet \( ABA \) where energies \( A \) and \( B \) are comparable. Let us refer to these spins by the labels 1 through 3, and similarly label the external neighboring spins as 0 and 4. Suppose spins 0, 2, and 4 are initially in state \( | \uparrow \rangle \). Since spin 0 remains far off resonance from 1, their interaction is effectively of the Ising form \( J_Z \sigma_0^z \sigma_1^z \). Similarly the interaction between 3 and 4 is \( J_Z \sigma_2^z \sigma_3^z \). Moreover, those external spins (having only an Ising interaction with their neighbors) are frozen in the \( | \uparrow \rangle \) state thus their interaction with the triplet reduces to \( J_Z \sigma_4^z \) and \( J_Z \sigma_2^z \), and the dynamics of the triplet are described by the Hamiltonian

\[
H_{\text{triplet}} = H_{\text{Zeeman}} + H_{\text{int}},
\]

\[
H_{\text{Zeeman}} = (A + J_Z) (\sigma_1^z + \sigma_2^z) + B \sigma_4^z,
\]

\[
H_{\text{int}} = \sum_{j=1,2} J_{XY} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + J_Z \sigma_2^z \sigma_4^z.
\]

In the following we will use \( a = A + J_Z \) (the effective Zeeman energy of spins 1 and 3) and we will define \( b = B \) for consistency. The Hamiltonian is easy to analyze; the states \( | \uparrow \uparrow \rangle \) and \( | \downarrow \downarrow \rangle \) of course remain eigenstates while the other states form two distinct subspaces. For the up subspace spanned by \( \{ | \uparrow \downarrow \rangle, | \uparrow \uparrow \rangle, | \uparrow \downarrow \rangle \} \) we have Hamiltonian and eigenvectors given by

\[
\tilde{H}_U = b \frac{1}{2} + 2J_{XY} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\Rightarrow |a\rangle_U = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\]

and

\[
| \pm \rangle_U = \begin{pmatrix} \frac{1}{2}(p \pm S_p) \\ 1 \end{pmatrix}
\]

With corresponding energies \( E_{aU} = b \), \( E_{aU} = b + J_{XY}(p \pm S_p) \). Here \( p = (a - b - J_Z) / J_{XY} \) and \( S_p = \sqrt{8 + p^2} \). Similarly for the complimentary down space \( \{ | \downarrow \downarrow \rangle, | \downarrow \uparrow \rangle, | \downarrow \downarrow \rangle \} \) we have

\[
\tilde{H}_D = -b \frac{1}{2} + 2J_{XY} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\Rightarrow |a\rangle_D = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\]

and

\[
| \pm \rangle_D = \begin{pmatrix} \frac{1}{2}(q \pm S_q) \\ 1 \end{pmatrix}
\]

With energies \( E_{aD} = -b \). \( E_{aD} = b + J_{XY}(p \pm S_p) \), where \( q = (a - b - J_Z) / J_{XY} \) and \( S_q = \sqrt{8 + q^2} \). Now, we know that the initial computational qubit states are

\[
|00\rangle = | \uparrow \downarrow \rangle \Leftrightarrow \text{composed of} \ |+\rangle_U \text{ and } | - \rangle_U,
\]

\[
|01\rangle = | \uparrow \uparrow \rangle \Leftrightarrow \text{composed of} \ |a\rangle_U, | + \rangle_U \text{ and } | - \rangle_U,
\]
\[ |10\rangle = \left| \uparrow \downarrow \right> \leftarrow \text{composed of } \left| a \right>_U, \left| + \right>_U, \text{ and } \left| - \right>_U, \]
\[ |11\rangle = \left| \uparrow \uparrow \right> \leftarrow \text{eigenstate.} \]

During the gate operation, the states (other than \(|11\rangle\)) will rotate within their subspaces. We must arrange to reviv-although there may be various detunings for which the reviv-
effectively switch off the exchange interaction in this case we see that
whether such a transform constitutes a useful gate depends on entanglement criteria as mentioned later. The times for which \(|00\rangle\) are determined by \(E^{u}_{f} - E^{l}_{u}\). The times at which a state, initially in the \(|01\rangle, |10\rangle\) plane, returns to that plane are determined by \(E^{u}_{f} - E^{l}_{u}\). Now the parameter which we can experimentally vary is the Zeeman detuning \(a - b\); although there may be various detunings for which the revivals coincide (which could be found numerically), there is one value that is immediately obvious by inspection: \(a - b = 0\) (corresponding to tuning the central barrier spin to \(A + J_2\)). In this case we see that \(p = q = -J_2/J_{XY}\), \(S_p = S_q = \sqrt{8((J_2/J_{XY})^2 + J_2^2))} \approx 2\). Thus both revivals coincide at time \(t_R = \pi h(8J_{XY}^2 - J_2)^{-1/2}\). At this instant, the transformation in the computational basis \(|00\rangle, |01\rangle, |10\rangle, |11\rangle\) is given by the following matrix (neglecting a global phase):
\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & iQc & Qc & 0 \\
0 & Qc & iQs & 0 \\
0 & 0 & 0 & W
\end{pmatrix}.
\]

Here \(Q = - \exp(i\phi)\), \(s/c = \sin/cos(\phi)\) and \(W = - \exp(-2i\phi)\) with \(\phi = (\pi/2)(8J_{XY}^2/J_2^2 + 1)^{-1/2}\). The phases in this matrix are with respect to the passive state of the device (i.e., if we had not tuned the triplet into resonance), under the assumption that the resonance was achieved by shifting the Zeeman energy of the central spin. (If in fact the Zeeman energies of the qubit-bearing spins were adjusted to achieve resonance, then we simply have the above matrix together with two trivial single qubit \(Z\) gates.) This transformation \(U\) is entangling, and is therefore adequate to construct a universal gate set when combined with single qubit gates [16]. Using the procedure described in Refs. [16,17] one can confirm that no more than four of these gates are required to form a controlled-\(\text{NOT}\) operation, for a wide range of \(J_Z\) including the \(J_Z = 0\) and \(J_Z = J_{XY}\) cases, which represent the \(XY\) interaction and the isotropic Heisenberg interaction, respectively. It is easier to appreciate the nature of the transform if we apply a couple of single-qubit \(Z\) rotations; defining
\[
Z(\theta) = \begin{pmatrix}
\exp(i\theta) & 0 \\
0 & \exp(-i\theta)
\end{pmatrix}
\]
then neglecting a global phase,

\[
Z(\phi)Z(\psi)U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -Q's & iQc & 0 \\
iQc & iQ's & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Here \(\psi = \pi/4\)\(1 - (8J_{XY}^2/J_2^2 + 1)^{-1/2}\) and \(Q' = Q^2\) while \(s/c\) are as before. Notice that for the \(J_Z = 0\) limit, i.e., the case of a pure \(XY\) interaction, then the primitive matrix \(U\) takes a particularly simple form [18].
\[
U_{p} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

using which one can construct a \(\text{CNOT}\) gate with only two applications, as shown in Fig. 1(b). In this limit, the dressed matrix (1) is recognizable as the “iswap” which has been studied in the context of an \(XY\) interaction between adjacent qubits [19]. Indeed, in the limit of a strict \(XY\) interaction, one might choose to abandon the barrier spin architecture completely, and adopt a trivial architecture in which qubits are adjacent (since the primary function of the barrier spins is to negate the effect of the residual Ising interaction, absent for the pure \(XY\) form).

Note the second form of gate presented in Ref. [1] can also be generalized to anisotropic Heisenberg interactions, although it does require some finite \(Z\) component since this is exploited to accumulate a phase during the gate operation.

III. NONABRUPT ZEEMAN SHIFTS

In Ref. [1] and in the above analysis, we consider an abrupt change from far off-resonance spins into resonance. This may be difficult to achieve in many otherwise promising implementations, therefore we now investigate the less physically demanding case of smooth switching. Figure 2 shows various profiles for the dynamically changing Zeeman...
energy of the central spin, given that the Zeeman energies of the outer spins are static [Fig. 2(a) corresponds to analytic treatment given above]. For numerical convenience we have built these switching profiles as piecewise combinations of analytic functions, as defined in the figure caption. In both cases [Figs. 2(b) and 2(c)] we fixed the time for the switching transition to the arbitrary choice \( \tau_3 = 1.25 \) and varied just a single parameter, the time for which the detuning is zero.

The values shown in the Fig. 2 provided a complete revival of the central spin for all qubit basis states, just as in the case of the abrupt transition. The specific transformation achieved in the qubit basis [i.e., the analogue of Eq. (1)] is of course different for these smooth switching profiles, but it remains strongly entangling and therefore equally suitable as a primitive two-qubit gate.

**IV. HIGHER DIMENSIONAL ARRAYS**

The analysis presented in the present paper has been phrased in terms of a one-dimensional array (a). However, the basic gate construction, involving two qubit-bearing spins and one barrier spin, can immediately be generalized to many geometries in either two, or three dimensions. In principle one can produce a suitable structure by taking any arrangement of qubit-bearing spins, and introducing a barrier spin between each (hitherto) adjacent pair. One possible measure of the efficiency of the implementation would be the ratio of qubit-bearing spins to total number of spins, which we can denote \( R_Q \). The value \( R_Q = \frac{1}{2} \) corresponds to the one-dimensional arrangement [Fig. 3(a)]. For a two, or higher, dimensional geometry at least some of the qubits must of course have three or more neighbors. If we restrict ourselves to considering regular structures in which every qubit has the same number of neighbors, then it is apparent that the highest possible value of \( R_Q \) is 2/5. Two arrangements which achieve this value are the hexagonal geometry Fig. 3(b), and the three-dimensional (3D) structure illustrated in Fig. 3(c).

In order to do better than this ratio it would be necessary for barrier spins to do double duty in the sense that each barrier could not be unique to a specific qubit pair. Figure 3(d) shows an example arrangement achieving \( R_Q = 3/5 \) in two-dimensions (2D). Note Fig. 3(d) is the compliment of Fig. 3(b), i.e., the qubit and barrier roles are reversed; similarly, one could reverse Fig. 3(c) for a 3D form. In such a structure, bringing a barrier into resonance with its neighbors would initiate a three-qubit gate process—to successfully complete the gate one would require the simultaneous revival of all qubit basis states at some subsequent moment. As the number of qubits involved increases, this will presumably quickly become infeasible (see Appendix B), but both the three qubit gate shown in Fig. 3(d), and a four qubit variant, do deserve analysis [23].

**V. CONCLUSION**

We have extended the results presented in Ref. [1] in several significant respects. The first, fundamental generalization is from a pure isotropic Heisenberg interaction to a more general anisotropic interaction, including the in-plane \( XY \) interaction as a special case. All the results presented here
incorporate this generality. We have provided a proof that an interaction of this general form tends to a simple Ising interaction in the limit of far off-resonance neighbors. We have presented an analysis of the basic gate of Ref. [1] in with this general interaction, and exhibited the resulting primitive two-qubit gate. In the special case of an XY interaction, we note that the gate has an especially simple form and we provided an explicit circuit for an efficient CNOT gate based on this primitive. We also consider the effect of a nonabrupt switching of the Zeeman energy: by numerical simulation we demonstrate that simply varying the duration of the on-resonance phase (while the switching time remains constant) allows one to achieve the necessary revival of the barrier spins, and therefore abrupt switching is not a requirement of the scheme. Finally we have remarked upon the simplicity of generalizing to two- and three-dimensional arrays, noting that the array geometry then determines the scheme’s cost in terms of the proportion of barrier spins.

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APPENDIX A: ANALYSIS OF HEISENBERG CHAIN WITH LARGE ZEEMAN DISCREPANCIES

Given the definition of $H = H_1 + H_2$ introduced in the main body of the paper, we can proceed to use the Trotter formula to write the time evolution $U(t)$ as

$$U(t) = [\exp(-i H_1 t/n) \exp(-i H_2 t/n)]^n \text{ as } n \to \infty. \quad (A1)$$

Now we will seek to move all $H_1$ terms to the right, thus separating the Ising and non-Ising parts. First we will introduce a generalization of $H_2$,

$$H_2^W = \frac{I_{XY}}{2} \sum_j \left(W_j \sigma_j^+ \sigma_{j+1}^- + W_j \sigma_j^- \sigma_{j+1}^+\right)$$

where the $W_j$ are any functions involving scalar constants and $\sigma_j^\pm$ for any $k,l$.

To reorder a pair of terms $\exp(-i H_1 \tau) \exp(-i H_2 W \tau)$, where we are writing $\tau=t/n$, we will exploit the following general observation for operators $A$ (unitary) and $B$:

$$A \exp(B) = A \exp(B) A^\dagger A = \exp(ABA^\dagger) A.$$

We will choose $B=-iH_2 W \tau$ and $A=\exp(-iH_1 \tau)$. Then the argument of the exponent that we must evaluate is $ABA^\dagger = -i \tau \exp(-iH_1 \tau) H_2^W \exp(i H_1 \tau)$.

Note first that since

$$[\sigma_j^z, \sigma_k^z] = 0, \quad [\sigma_j^x, \sigma_k^y] = 0, \quad [\sigma_j^y, \sigma_k^x] = 0$$

we can write the following:

$$\exp(-i H_1 \tau) = \left( \prod_j \exp(-iB_j \sigma_j^y) \right) \left( \prod_j \exp(-iJ_z \sigma_j^z \sigma_{j+1}^z) \right)$$

and in fact we can reorder the product terms as we wish. Noticing that

$$\sigma_j^z \sigma_j^\pm = \pm \sigma_j^\pm, \quad \sigma_j^\pm \sigma_j^z = \mp \sigma_j^z$$

and

$$\exp(i C \sigma_j^z) = \cos(C) I + i \sin(C) \sigma_j^z,$$

$$\exp(i C \sigma_j^z \sigma_{j+1}^z) = \cos(C) I + i \sin(C) \sigma_j^z \sigma_{j+1}^z$$

for any constant $C$, we conclude that

$$\exp(i C \sigma_j^z) \exp(-i C \sigma_j^z) = \exp(\pm 2i C) \sigma_j^z$$

(assuming $k \neq j$). We can then write the following:

$$\exp(-i \tau B \sigma_j^y) (W_j \sigma_j^+ \sigma_{j+1}^- + W_j \sigma_j^- \sigma_{j+1}^+) \exp(i \tau B \sigma_j^y)$$

$$= e^{-2i \tau B_j} W_j \sigma_j^+ \sigma_{j+1}^- + e^{2i \tau B_j} W_j \sigma_j^- \sigma_{j+1}^+ \quad \text{if } k = j$$

$$= e^{-2i \tau B_j} W_j \sigma_j^+ \sigma_{j+1}^- + e^{-2i \tau B_j} W_j \sigma_j^+ \sigma_{j+1}^+ \quad \text{if } k = j + 1$$

$$= W_j \sigma_j^+ \sigma_{j+1}^- + W_j \sigma_j^+ \sigma_{j+1}^+ \quad \text{otherwise}.$$
stans and $\sigma^Z_k$ for any/all $k$), we can just repeat the argument to commute all terms $\exp(iH_1\tau)$ to the far right. The term originally identified as the $m$th element $\exp(iH_2\tau)$ in the Trotter expansion (A1) will have $m$ terms “$\exp(-iH_1\tau)$” pass through it, and will thus accumulate a final $Q(m) = \exp\{im(t/n)[(\Delta+2J_2(\sigma^Z_{j+2}-\sigma^Z_{j-1})]\}$. So the exact expression for the time evolution finally becomes

$$U(t) = R(t)\exp(-iH_1t)$$

where

$$R = \left\{ \prod_{m=1}^{n=\infty} \exp\left[ -\frac{it}{n} H_R\left( \frac{mt}{n} \right) \right] \right\}$$

$$H_R(\eta) = \frac{J_{XX}}{2} \sum_j X_j(\eta)\sigma^x_j(\sigma^x_{j+1} + X_j(\eta)\sigma^z_j\sigma^z_{j+1}$$

with

$$X_j(\eta) = \exp[i\eta(\Delta + 2J_2(\sigma^Z_{j+2}-\sigma^Z_{j-1})]]$$

as quoted in the main text.

The right-hand term in $U(t)$ is the pure Ising chain evolution we seek, but the subsequent residual operator $R$ is more complex. We would like to show that it tends to unity as $\delta_j = J_{XX}/\Delta_j \to 0$ for all $j$. Now we cannot simply integrate the terms in the product $R$ since they do not commute, and thus we cannot immediately gather the elements with a $t/n$ coefficient. Therefore we proceed by making the expansion

$$\prod_{m=1}^{\infty} \exp\left[ -\frac{it}{n} H_R\left( \frac{mt}{n} \right) \right] = \prod_{m=1}^{\infty} 1 - \frac{1}{2}\left[ -\frac{it}{n} H_R\left( \frac{mt}{n} \right) \right]^2 + \cdots,$$

as $n \to \infty$. We cannot truncate this series since $H_R(\cdot)$ is not small, but we will seek to gather and sum all terms of given order in $H_R$. We introduce

$$g_a = \frac{J_{XY}}{2} \int d\eta X_j(\eta) = -\frac{i\delta}{2} x_jX_j(\eta),$$

where

$$x_j = \rho_i[1 - 8\alpha^2\delta_j(1 + \sigma^z_j\sigma^z_{j+2})]\left[1 - 2\alpha\delta_j(\sigma^z_{j+2} - \sigma^z_{j-1})\right]$$

$$+ \rho_{ij}$$

with the approximation holding in the limit that all $\delta_j = J_{XY}/\Delta_i \ll 1$. Note that $\rho_{ij}$ and thus $x_j$, is a modest ratio in our periodic chains [e.g., for an $ABAB\cdots$ chain $\rho_{ij} = (-1)^j$; for an $ABCABC\cdots$ chain $\rho_{ij}$ might run 1, 1, $-\frac{1}{2}$, 1, $-\frac{1}{2}$, etc. 2 say]. We can write the indefinite integral

$$\int d\eta H_R(\eta) = i\frac{\delta}{2} \sum_j x_j[1 - X_j(\eta)\sigma^x_j\sigma^x_{j+1} + X_j(\eta)\sigma^z_j\sigma^z_{j+1}$$

$$= \frac{i\delta}{2} K(\eta).$$

Using $K(\cdot)$ defined above we can write

$$g_a = \frac{a}{2}[K(t) - K(at/n)].$$

Now returning to the expansion, the lowest order in $H_R$ is of course 1, and the sum of all terms of first order in $H_R$ is precisely $-i\delta = -\frac{a}{2}\delta[[K(t) - K(0)]]$. Thus so far we are seeing the anticipated behavior, the residual part of the dynamics, after the Ising-type behavior is allowed for, appears to vanish with $\delta$. However, since we are using an expansion in $H_R(\cdot)$, where $H_R(\cdot)$ is not small, we should evaluate and sum the higher terms. Let us use the symbol $S_t$ to represent the sum of terms of order $H_R(t)$ within our expansion; then we have already found $S_1 = \frac{a}{2}[K(t) - K(0)]$, and

$$S_2 = -\left( \frac{a}{2} \right)^2 \sum_{m=1}^{\infty} \left\{ \frac{1}{2} H_R(mt/n)^2 + H_R(mt/n) \sum_{p>m} H_R(p\ell/n) \right\},$$

now the factor $(t/n)^2$ causes the first term here to vanish in the limit $n \to \infty$, since it contains only $n$ terms each of order $(H_R(t) \sim J_{XY})$. For the second term

$$S_2 = \frac{a}{2} \int_0^t H_R(\eta)\left( \int_\eta^t H_R(\eta) d\eta \right) d\eta,$$

but the inner integral is given by (5) so that

$$S_2 = -\frac{a}{2} \int_0^t H_R(\eta)[K(t) - K(\eta)] d\eta.$$
in the phase we would apply a factor of order unity—but we can never introduce a factor of Δ.

Generalizing this observation we can consider $S_N$. This involves terms of the form $H_R(m_1t/n)H_R(m_2t/n)\cdots H_R(m_Nt/n)$ for some set of integers $m_1 \gg m_2 \gg \cdots \gg m_N$. By the same reasoning above, we can neglect terms where two or more of the $m_i$ are the same value, since they collectively constitute a negligible portion $1/n$ of the sum as $n \to \infty$. Then we find

$$S_N = (-i)^N \int_0^t H_R(\xi_1) \int_{\xi_1}^t H_R(\xi_2) \int_{\xi_2}^t \cdots \times \int_{\xi_{N-1}}^t H_R(\xi_N)d\xi_1 d\xi_2 \cdots d\xi_N$$

$$= \frac{(i)^{N-1}}{2} \delta \int_0^t H_R(\xi_1) \int_{\xi_1}^t H_R(\xi_2) \int_{\xi_2}^t \cdots \times \int_{\xi_{N-2}}^t [K(t) - K(\xi_{N-1})]d\xi_1 d\xi_2 \cdots d\xi_{N-1}.$$

And as before we can argue that although the remaining $N - 1$ integrals may produce additional factors of $1/\Delta$, they cannot absorb any. Thus the factor $\delta$ will remain and we can conclude that all terms in the expansion $S_n (n \gg 1)$ are of order $\delta$ or less. Therefore the time evolution operator is

$$U(t) = [1 - \delta P(t)]\exp(-iH_Rt)$$

for some finite operator $P(t)$ whose magnitude does not increase with $\Delta$. This is the result presented in the main body of the paper.

**APPENDIX B: REGARDING REVIVALS**

In the discussion of two- and three-dimensional arrays, we stated that it will be difficult to achieve the crucial simultaneous revivals for multiqubit gates involving more than a few qubits. Of course, one can observe that if we choose any detuning $A - B$ for which the revival periods of the various qubit basis states are related by irrational factors (i.e., the general case), then there will eventually be a complete revival to any desired accuracy (although never perfect). However one would typically need to wait an extremely long time for the level of precision required for QC and therefore this type of revival is not a practical choice. Instead we seek to arrange rapid revivals by looking for values of the detuning (and potentially, other parameters) such that the various revival periods are all related by small rational factors. Fulfilling this condition may become unfeasible as the number of qubits increases.

[18] A very recent online preprint, which restricts itself to the XY limit, has also exhibited this matrix—see Ref. [3].
[20] The function here is a symmetric two part composite. Defining $\tau = t - t_i$, the Zeeman shift is introduced (switched on) in the period $0 < \tau < t_\Delta$ by a function of the form $-\sin^4(\phi \tau)$ for $\tau < t_\Delta/2$, and $\sin^4(\phi (\tau-t_\Delta)) - 1$ for $\tau > t_\Delta/2$, where $\phi = 2 \arctan(2^{-1/4}/t_\Delta)$.
[21] For each profile we performed a series of numerical simulations, manually adjusting duration until we obtained revivals that were perfect to within an error probability of about 1 part in $10^6$, apparently one could continue to refine the value arbitrarily.
[22] The integral is valid provided $4J_2 \neq \Delta_j \forall \Delta_j$, which is the general case and moreover typically if $J_2$ is non-zero it will be of order $J_{XY}$ (and of course we are interested in the $J_{XY} \leq \Delta_j$ limit throughout).